Visualizing Cubic Algebraic Surfaces

A Thesis Presented to

The Faculty of the Mathematics Program

California State University Channel Islands

In Partial Fulfillment

of the Requirements for the Degree

Masters of Science

by

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July 2011

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ACKNOWLEDGEMENTS

Thank you to my advisor, Dr. Ivona Grzegorczyk, who has spent more than two years and countless hours working on this project with me. Her constant encouragement and guidance was the driving force behind this research.

Thank you to Dr. Cindy Wyels and Dr. Brian Sittinger, for being a part of my thesis committee. Thank you to the math department as a whole; they have been incredibly supportive throughout this process. CSU Channel Islands has given me the best education I could ask for, throughout both my undergraduate and master's degrees; and the mathematics faculty is the cornerstone of this experience.

Most of all, thank you to my family. Thank you to Mom and Dad, for their support and advice throughout my academic journey. Thank you to Drew, for his love and reassurance. Thank you to my sister, my grandmas, my friends…thank you to everyone who had faith in me. We did it!

ABSTRACT

A cubic surface is defined as the set of zeroes of a homogeneous polynomial f of degree three in three-dimensional real projective space given by

$$S = \{(x : y : z : w) \in \mathbf{P}^3(\mathbf{R}) \mid f(x, y, z, w) = 0\}.$$

The geometric classification of these objects remains an unsolved problem. Ideally, such a classification would incorporate information about the notable geometric properties of each surface, yet be general enough to encompass all cubic surfaces succinctly. Using new visualization tools, we review and develop methods to identify several of these properties; namely, the symmetry exhibited by a surface, the real valued lines on a surface, and the presence and number of singular points on a surface. We also experiment with the effect that deformation of the surface has on these properties, with the goal of studying their stability under such deformation.

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1 Introduction

Algebraic surfaces in n-dimensional real projective space $\mathbf{P}^n(\mathbf{R})$ are the collections of points satisfying a finite number of polynomial equations of the form $p(a_1, a_2, \ldots, a_m) = 0$. Even in the case of degree three surfaces in real projective space $\mathbf{P}^3(\mathbf{R})$, finding a comprehensive geometric classification still remains an open problem, even though extensive work has been conducted in this direction over the past two hundred years. In order to address the situation, we study the unique geometric properties that each surface exhibits and use new computer-based tools to visualize them. In particular, we study the presence of lines on a surface, singular points, and the isometric symmetries exhibited by that surface. Furthermore, we experiment with deformations of these surfaces, and analyze the effect of the deformation on these properties.

In this paper, we will present a review of the history of cubic surfaces and the progress already made towards classifying them. We describe the properties of a "good" geometric classification through a careful analysis of quadratic surfaces. We then provide an in-depth analysis of methods for meaningfully describing the geometric properties of a cubic surface. This includes the presentation of an original method for finding the number of real valued lines on a cubic surface and their explicit parametric equations. We also introduce an original application of level curves to analyze geometric properties of surfaces. These methods, combined with more traditional approaches to feature analysis, allow us to meaningfully analyze and quantify certain geometric properties of cubic surfaces, given their implicit-form equations. Finally, we provide several examples of

smooth deformations of surfaces and study the effect of these deformations on these geometric properties.

2 Preliminary Material

2.1 Restrictions

For this study, we restrict ourselves to real three-dimensional projective space $\mathbf{P}^3(\mathbf{R})$. In general, projective space can be defined for any dimension, and over any field. However, our restriction implies that we study real-valued algebraic surfaces, which will be explicitly defined in Section 2.3. We also restrict our interest to real-valued properties; i.e., lines given in $\mathbf{P}^3(\mathbf{R})$, singularities at real-valued coordinate locations, etc. Algebraic surfaces may also be defined over $\mathbf{P}^3(\mathbf{C})$. Moreover, cubic surfaces in $\mathbf{P}^3(\mathbf{R})$ may contain complex features—i.e., a pair of lines given by complex conjugates. Given our goal of visualization, we omit analysis of such features, as they cannot be presented in real space. Throughout this paper, the reader should assume these restrictions regardless of whether they are explicitly stated in that section.

2.2 Visualizing Real Projective Space

We begin by defining three-dimensional real projective space $\mathbf{P}^3(\mathbf{R})$.

> Definition 1 $\mathbf{P}^3(\mathbf{R})$ is defined as $R^4 \setminus \{0\} / \sim$, where $\sim$ is the equivalence relation $(x,y,z,w) \sim (\lambda x, \lambda y, \lambda z, \lambda w)$ such that λ is an arbitrary non-zero real constant.

In other words, it is the set of classes of all lines on $\mathbf{R}^4$ passing through the origin.

Important properties of $\mathbf{P}^3(\mathbf{R})$ are as follows:

1. $\mathbf{P}^3(\mathbf{R})$ is a homogeneous space. Therefore, the notation for a point is given by projective homogeneous coordinates $[x: y: z: w]$.
2. $\mathbf{P}^3(\mathbf{R})$ is a compact topological space.
3. $\mathbf{R}^3 \subset \mathbf{P}^3(\mathbf{R})$.
4. When $w = 1$, the Zariski open subset is isomorphic to $\mathbf{R}^3$. There are four distinguished subsets given by $x=1$, $y=1$, $z=1$, $w=1$ isomorphic to $\mathbf{R}^3$. They cover $\mathbf{P}^3(\mathbf{R})$.

5. $w = 0$ defines $\mathbf{P}^2(\mathbf{R})$, which is a projective line in $\mathbf{P}^3(\mathbf{R})$ (a plane at "infinity" for the set w=1).

Toplogically, we can describe $\mathbf{P}^3(\mathbf{R})$ as a "compactification" of $\mathbf{R}^3$. To do so, visualize $\mathbf{R}^3$ as a ball. The points on the boundary of the ball are the points at infinity. The points in the interior of the ball are "normal" points that can be given by (x_0,y_0,z_0), where $x_0,y_0,z_0 \in \mathbf{R}$. Points on the boundary are then "glued" together when they lie on the same line passing through (0,0,0,0). This changes lines into curves that are completely contained within the space.

Example 1

Consider the following visualization of $\mathbf{P}^2(\mathbf{R})$ given in Figure 1. We have a plane (namely, $\mathbf{R}^2$) bounded by the points at infinity. Let a be a line on the plane. Take the "endpoints" of the line, namely the points at infinity, and "glue" them together. The plane "bends" until the two points touch. $\mathbf{P}^2(\mathbf{R})$ is precisely made up of all such projective lines. Figure 1 shows $\mathbf{R}^2$ as a hemisphere, as well as two distinct lines a and b, and the manner in which their endpoints will be glued.

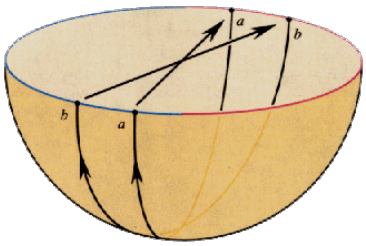


Figure 1. Visualizing Projective Space. Image from [13].

Curves in $\mathbf{R}^2$ behave similarly. For example, all of the nondegenerate conic sections (parabolas, hyperbolas, and ellipses) are topologically equivalent in $\mathbf{P}^2(\mathbf{R})$, because the points at infinity for the parabolas and hyperbolas have been connected, forming ellipses.

2.3 Cubic Algebraic Surfaces

We now define a cubic surface in $\mathbf{P}^3(\mathbf{R})$.

Definition 2 A cubic surface $S \subseteq \mathbf{P}^3(\mathbf{R})$ is defined as the set of zeroes of a single homogenous polynomial f of degree three in $\mathbf{P}^3(\mathbf{R})$.

$$S = \{(x: y: z: w) \in \mathbf{P}^3(\mathbf{R}) \mid f(x, y, z, w) = 0\}.$$

These surfaces inherit the properties of the space; namely, they are compact. Note that since the space is homogenous, each term in the equation of a cubic surface is necessarily of degree three.

2.3.1 Geometric Properties

Throughout this paper, we will refer to certain geometric properties of a cubic surface. These include the following.

Definition 3. A *singularity* on a cubic surface is a point on the surface for which the tangent plane is not well defined, i.e., the gradient is equal to 0.

Example 2
Consider the surface given by $x^3 + y^3 + z^3 + xyz = 0$.

$$\frac{\partial f}{\partial x} = 3x^2 + yz$$

$$\frac{\partial f}{\partial y} = 3y^2 + xz$$

$$\frac{\partial f}{\partial z} = 3z^2 + xy$$

Consider the point $p = (0,0,0) \in S$.

$$\nabla f(p) = (0, 0, 0)$$

Therefore, there is a singularity on the surface at the point (0,0,0). See Figure 2.

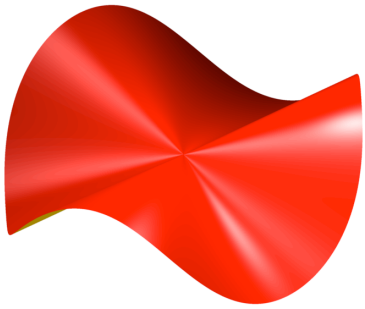


Figure 2. A singular surface. Image generated using [4].

Definition 4. A *line on a cubic surface* is a line in $\mathbf{P}^3(\mathbf{R})$ given by a set of homogenous parametric equations in $[x: y: z: w]$ that is completely contained by the surface. The equation of the line on a surface is a solution to the polynomial defining the surface itself.

Example 3
There is a line on the surface from Example 2 given by (t, $-t$, 0, s).

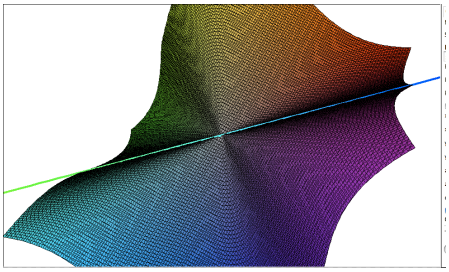


Figure 3. Line on a surface.
Image generated using [12].

Definition 5. The *symmetry* of a surface S is given by set of reflections and rotations on that surface. These constitute an algebraic group, denoted Aut(S). These are proper subgroups of the symmetry of the space $\mathbf{P}^3(\mathbf{R})$.

Example 4
The surface from Example 2 is symmetric under permutation of three coordinates x, y, and z. This corresponds to the group S_3, the group of permutations on three elements. In general, the symmetry of a surface with symmetry group S_n is given by the subgroup of S_n that consists of orientation-preserving symmetries (i.e., reflections and rotations).

The properties above are key to a geometric description of a cubic surface. Each of these properties can be visualized in real space. Moreover, these properties are identifiable for any cubic surface. While a cubic surface is not necessarily uniquely determined by its properties, they serve as a meaningful way to describe the geometry of that surface. We will show that these properties are also *not* invariant under geometric deformation, even those that are given by varying a single constant in the defining equation. We will explore this in depth in Chapter 6.

2.3.2 Cubic surfaces of interest

In this paper, we demonstrate our methods of analysis on several cubic surfaces with interesting geometric properties. These surfaces are as follows:

Fermat's Cubic Surface

$$x^3 + y^3 + z^3 + w^3 = 0$$

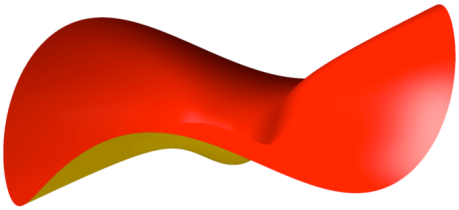


Figure 4. Fermat's cubic surface.
Image generated using [4].

Clebsch Diagonal Surface

$$x^3 + y^3 + z^3 + w^3 = (x + y + z + w)^3$$

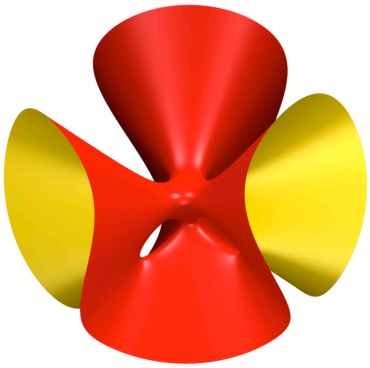


Figure 5. Clebsch diagonal surface.
Image generated using [4].

Ding-dong Surface

$$z^3 + x^2w + y^2w - z^2w = 0$$

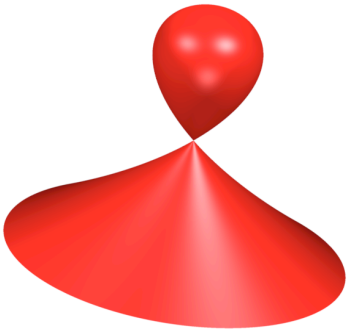


Figure 6. Ding-dong surface.
Image generated using [4].

Cayley's Nodal Surface

$$xyz + xyw + xzw + yzw = 0$$

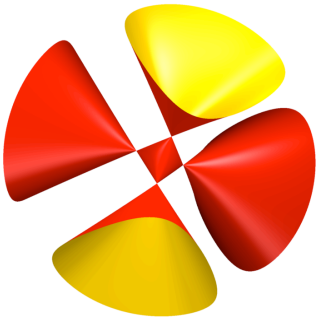


Figure 7. Cayley's nodal surface.
Image generated using [4].

2.4 Computer-Aided Visualization Tools

Given that the surfaces for our study are embedded in $\mathbf{P}^3(\mathbf{R})$, visualization presents a major challenge. Intrinsically, we can only visualize objects in $\mathbf{R}^3$. One of the goals of our project is to identify geometric features of the surface from the equation and provide affine equations in $\mathbf{R}^3$ that produce surfaces with these features. By restricting ourselves to subsets of $\mathbf{P}^3(\mathbf{R})$ isomorphic to $\mathbf{R}^3$, we lose the information contained in the points at infinity, and so these must be checked separately for interesting geometric properties. We

will give more information about this in the section on finding lines. Notice that we often simply use the set given by $w=1$. This is because of the equivalence relation ~; we choose 1 as the representation of the equivalence class of nonzero real constants, i.e., we divide by any nonzero value of w. For example, $(2,3,4,5) \sim \left(\frac{2}{5},\frac{3}{5},\frac{4}{5},1\right)$. However, it is necessary in some cases to choose a different value for w in order to view the interesting geometric properties of the surface.

Once we have an affine equation of a cubic surface, we can visualize it using computer programs. A few examples of computer algebra systems that do so are Maple, Mathematica, and Matlab. In this paper, we make extensive use of Surfer, a free open-source program available through the Fink project. Surfer is based on the program Surf and was developed for the exposition IMAGINARY, which was conceived by the *Mathematisches Forschungsinstitut Oberwolfach* for the German "Year of Mathematics 2008" **[4]**. The user simply inputs an implicit polynomial equation of three variables, and the visualized surfaces are given as the zeroes of those polynomials. Therefore, once we have an affine representation of a cubic surface, this program makes it extremely easy to visualize that surface. The user may then input parameters that allow for a smooth deformation of the surface, and even create a movie using screen-shots.

This recent technology addresses one of the stumbling blocks for cubic surfaces (indeed, algebraic surfaces in general). Historically, mathematicians could only “imagine” the surface based on a convoluted analysis of its equation. Using this technology, we can readily visualize any surface and identify the presence of certain geometric properties that

can then be verified through various methods of analysis. Indeed, there are several geometric properties that are obvious from visualization; we discuss these briefly in Section 5.6. In this paper, we focus on geometric properties that we can visualize, but are not trivial to identify.

3 Historical and Literature Review

3.1 Historical Background

Historically, the first interesting result for cubic surfaces was recorded in 1849 in correspondence between George Salmon and Arthur Cayley on the number of straight lines on a cubic surface. This led to the following theorem in 1865.

> Cayley-Salmon Theorem: There can be at most 27 real valued straight lines on any smooth cubic surface **[19]**.

In 1858, Ludwig Schlafli was the first to classify cubic surfaces with respect to the number of real straight lines and tritangent planes on them, identifying what he referred to as five "types" and 23 "species" of surfaces. He published these results in 1863 **[1]**. Cayley also presented Schlafli's classification and added further investigations of his own shortly thereafter **[19]**. However, this classification is only useful when considering nodal surfaces—every smooth cubic surface is considered to be of the same class and species.

These publications generated wider interest in cubic surfaces. For example, after learning of the Cayley-Salmon theorem, Steiner published an article that allowed purely geometrical treatment of cubic surfaces. In 1866, Cremona published *Memoire de géometrie pure sur les surfaces du troisième ordre*. In this memoir, he established many of the properties that had only been stated by Steiner **[17]**. The study of geometric properties led to interest in visualizing these surfaces.

In 1869, at Clebsch's suggestion, Christian Weiner constructed plaster of Paris models of cubic surfaces, which were then exhibited at several World Fairs **[16]**. Other mathematicians who made notable contributions to the construction and characterization of cubic surfaces were Sylvester, Klein, and LePaige **[17]**.

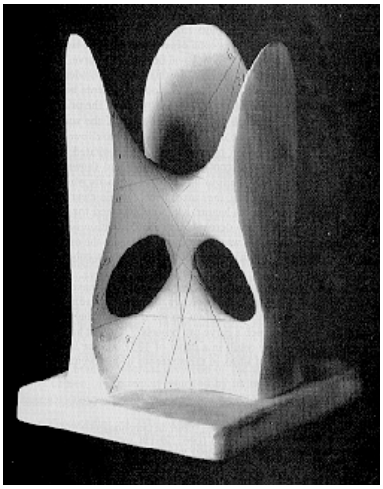


Figure 8. Photograph of Clebsch model. Image from [16].

One of the last mathematicians to publish significant results during this time period was Gino Fano, who worked with Klein to give an Italian translation of Klein's 1872 *Erlanger Program* (translation published 1893). This program described geometry as the study of the properties of a space that are invariant under a group of transformations. In the context of cubic surfaces, this amounted to a study of the symmetries of a surface **[17]**. Interest in cubic surfaces declined by about 1900, possibly due to *Hilbert's Programme,* a collection of 23 unsolved problems posed as a challenge to the mathematical community. This set the course for much of the research done in the 20^{th} century, and cubic surfaces fell out of fashion for many years. In fact, the subject was not widely revisited until the 1960s, when new techniques developed for differential geometry sparked a rebirth of interest in algebraic surfaces. We review several recent papers that contributed significantly to our work.

3.2 Recent Results

In 1979, Bruce and Wall published their paper "On the Classification of Cubic Surfaces" **[2]**. This provided a comprehensive, complete classification of all of the isolated singularities that are possible on cubic surfaces. An isolated singularity is a singularity

such that there is no other singularity in a (small) epsilon-neighborhood. Bruce and Wall define 8 types of singularities. For each one, they give its symbol, normal form, and Coxeter diagram **[2]**. We provide a short summary:

Conic Node	
A1	Singular point locally looks like that of a cone.
Binode	
A2	Two tangent planes are defined at the singularity, in various positions.
A3	
A4	
A5	
Unode	
D4	Two tangent planes coincide at the singularity.
D5	
Triple point	
E6	Simple elliptic singularity (cusp).

Table 1. Singularity classification [2].

In this paper, we consider these "non-degenerate" singularities; it is also possible to have a line of singular points if a surface intersects itself in some way. For example, the surface $xyz = 0$ is composed of three intersecting planes forming lines of singularities along the canonical axes.

In 1987, Knörrer and Miller published a topological classification for real cubic surfaces in their paper, "*Topologische Typen reeller kubischer Flächen*" **[8]**. Although this classification is complete, it is a broad classification due to the nature of topology. Topologically speaking, a plane and a saddle surface are indistinguishable. Yet, they are very different when considering geometric properties. Similarly, geometric properties of cubic surfaces are not always identified in the topological classification.

In 2006, Holzer and Labs published "Illustrating the Classification of Real Cubic Surfaces" in which they provide visualizations in $\mathbf{R}^3$ and equations in $\mathbf{P}^3(\mathbf{R})$ for all cubic surfaces given in Knörrer and Miller's classification that have only rational double points as singularities **[11]**. This resulted in a cubic surface "gallery" of 45 types, which can be accessed on their web site http://www.cubics.algebraicsurface.net **[10]**. This web site also provides several movies of deformations of these cubic surfaces and the effect of the deformations on their lines. It should be noted that the authors used an earlier version of Surfer **[4]** as well.

Polo-Blanco & Top's 2009 paper, "A remark on parameterizing nonsingular cubic surfaces" was also important for this research. Within the context of finding parameterizations of cubic surfaces, the authors present an algorithm for finding lines on cubic surfaces as well **[17]**. A detailed explanation of this paper's application to our research is given in Section 5.1.

4 Motivation and Goals for Geometric Classification

Ideally, a geometric classification for cubic surfaces would have the following properties:

1. Each type of surface is uniquely determined by a set of geometric properties;
2. These geometric properties can be meaningfully described;
3. The classification is comprehensive—any cubic surface falls into one of the classes;
4. The classification is not cumbersome (i.e., there are a relatively small number of simple classes);
5. The classification is not overly broad (i.e., surfaces with different geometric properties do not appear in the same class).

It should be noted that while Schlafli's 1863 paper describes a quite comprehensive classification, its main shortcoming is that all smooth cubic surfaces are considered to be of the same type. Given the many possibilities for lines and symmetry, this particular type is overly broad. Conversely, some of the types are quite specific. For example, Cayley showed that there is only one cubic surface up to projective isomorphism with four isolated singular points **[17]**. In Schlafli's classification, this is given as type 16 of 23.

The motivation for our problem comes from the fact that such a classification *does* exist for quadratic surfaces in $\mathbf{P}^3(\mathbf{R})$. There are 17 types of quadratic surfaces (three of which are degenerate); and each can be described in terms of a set of geometric properties **[9]**. In addition, each surface is uniquely given by a canonical equation in $\mathbf{R}^3$.

> Definition 6. A quadratic surface in $\mathbf{P}^3(\mathbf{R})$ is given by a single degree two homogenous polynomial equation
> $f(x,y,z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xz + a_5xy + a_6yz + a_7xw + a_8yw + a_9zw + a_{10}w^2 = 0$
> where a_i is any real number, not all 0.

Assuming $w = 1$ for visualization, the 17 types of surfaces are shown below in $\mathbf{R}^3$.

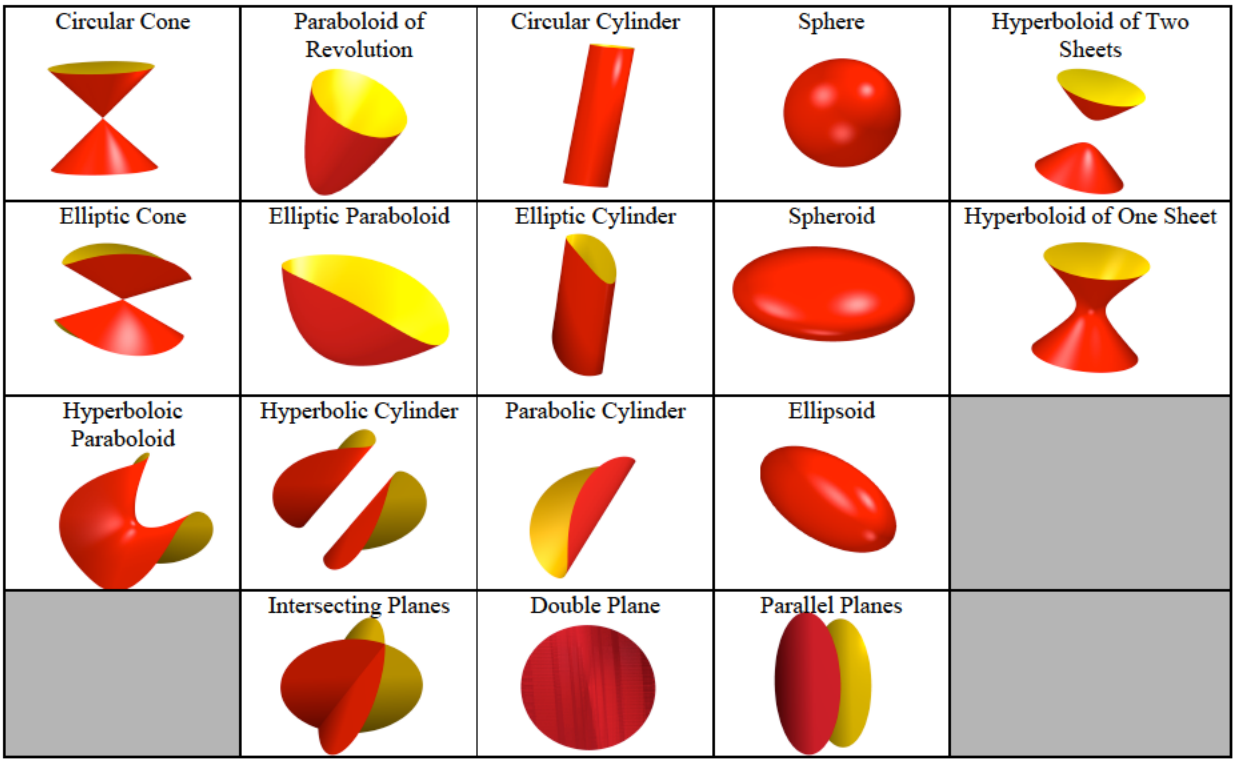


Figure 9. Quadratic surfaces. Images generated using [4].

We can explore the geometric properties of these surfaces. For example, is the surface ruled? How many pieces does it have? Are there any singular points? Is it bounded? What kind of symmetries does it have?

Example 5

In this example, we analyze the geometric properties of the circular cone shown in Figure 9. Precise definitions of the terms used in this example may be found in Appendix I. The canonical-form equation of the circular cone is $x^2 + y^2 = az^2$.

Symmetry:

- Reflective symmetry over planes given by $z = 0$ and any plane containing the z-axis.
- Rotational symmetry by any π-measurable rotation about z-axis, as well as rotations by π about the x and y axes.
- No translational symmetry.

Not bounded.

Not smooth: there exists a singularity at (0, 0, 0, 1). It is singularity type A_1.

The number of disconnected "pieces" is 1.

No self-intersections possible.

No saddle points or maximums/minimums in any direction.
Ruled
Curves lying on the cone include circles, as well as the conic sections.

For an in-depth analysis of each of these surfaces in terms of their geometric properties, see Appendix I. In Appendix I, we explore many geometric properties of these surfaces. However, for the purposes of this paper, we explore only the three aforementioned geometric properties for cubics. It is indeed possible that other properties could be important to a comprehensive classification. However, some of these properties are trivial upon visualization. Others are simply not applicable (i.e., ruled cubic surfaces are necessarily degenerate by the Cayley-Salmon theorem). For these reasons, we restrict our study to finding a meaningful way to quantify the three geometric properties mentioned earlier.

5 Geometric Properties of Cubic Surfaces

In this chapter, we explore geometric properties of a cubic surface, given its equation in $\mathbf{P}^3(\mathbf{R})$. Original work in this chapter includes a comprehensive method for finding lines on a cubic surface and giving their parametric equations, as well as some interesting applications of level curves to the surface to describe its symmetry.

5.1 Lines on a Cubic Surface

Definition 7. In $\mathbf{P}^3(\mathbf{R})$, a line on a cubic surface is given by a parametric homogenous equation of the form $\left(a_0 s + a_1 t,\ b_0 s + b_1 t,\ c_0 s + c_1 t,\ d_0 s + d_1 t\right)$, where $a_i,\ b_i,\ c_i,\ d_i \in R$ and $a_1,\ b_1,\ c_1,\ d_1$ are not all zero.

In this section, we present methods of solution to the following problems:

Given an equation for a cubic algebraic surface, identify:

1) The number of real valued lines on the surface.
2) Explicit parametric equations for these lines.

Remark. Holzer and Labs **[6]** give examples of real cubic surfaces with real valued lines, as well as the polynomial equations of the surfaces. They restrict their study to cubic surfaces with rational double points only (the 45 types of surfaces with this property classified topologically by Knörrer and Miller **[8]**). They have identified equations for real valued cubic surfaces with 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 21, 24, and 27 real valued lines. However, the parametric equations for these lines are not given. It is unknown whether real valued cubic surfaces exist with 13, 14, 17, 18, 19, 20, 22, 23, 25, and 26 lines lying on them.

If one can visualize the surface, or analyze its equation, the easiest method of finding lines on it is to simply guess-and-check. We graph the equation of a cubic surface and inspect it for possible lines. We estimate the coefficients in the parametric line equation. If $f = 0$ when evaluated by the equation of the line, the line is on the surface. Although this method is by no means comprehensive or efficient, it is very simple. This can be helpful as a starting point for finding lines that are easy to identify from the equation, but perhaps not as obvious from the visualization of the surface.

For example, one of the lines on the cubic surface given by $x^3 - y^3 + xzw + yzw = 0$ is given by $(t, t, 0, s)$. We show the affine representation in $\mathbf{R}^3$ by allowing $w = 1$ in Figure 8. In this setting, the equation for the surface is $x^3 - y^3 + xz + yz = 0$ and the equation for the line is $(t, t, 0)$.

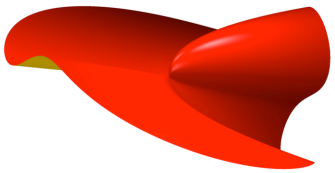


Figure 10. Cubic surface with a simple line. Image generated using [4].

Remark. Over the years, there have been dozens of authors that provide examples of lines on surfaces. Notably, Holzer & Labs' gallery of cubic surfaces includes lines in the images provided for each of their 45 surfaces **[6]**. However, it is rare that the author gives the equation for the lines, and even more rare that the author includes a systemic explanation for how they were found. Because of this, we focused on one paper in particular that included this information.

In their 2009 paper, Polo-Blanco & Top published a systematic method for finding lines on a cubic surface. This is given in the larger context of the study of parameterizing cubic surfaces. Below is a summary of their algorithm given in the paper **[18]**.

> In order to find a line on the surface one can proceed as follows. Any line $\ell \subset P_3$ is of course determined by two points (x_1, x_2, x_3, x_4) on it. To find the lines on a given cubic surface with equation $f(x, y, z, w) = 0$, first look for lines in the plane given by $x = 0$. This means, look for linear factors of $f(0, y, z, w)$. A line not in the plane $x = 0$ contains many points $(1, x_2, x_3, x_4)$. Next, for any parameter a, look for lines in the plane given by $y = ax$, i.e., find all a such that $f(x, ax, z, w)$ contains a linear factor, and give these factors. Any line not found so far, contains two points $(1, 0, a, b)$ and $(0, 1, c, d)$. Such lines are obtained using the following Maple program.

```
with(PolynomialTools):with(Groebner):
# select a cubic surface
F := x3 + y3 + z3 + w3:
# search for lines of the form (1 : t : a +ct : b +dt)
rule:={x = 1, y =t, z =a +c * t, w =b +d * t}:
F1a:= expand(subs(rule,F)):
F2a:= collect(F1a,t):
vect:=CoefficientVector(F2a,t):
F3a:=[vect[1],vect[2],vect[3],vect[4]]:
G1a:=gbasis(F3a,plex(a,b,c,d)):
sol:=solve({seq(G1a[i],i = 1 . . . 4)},{a,b,c,d});
```

In Example 6 in Section 5.2, we present a modified, commented version of the program shown above. After presenting this program, Polo-Blanco and Top go on to develop a method for parameterization of a cubic surface based on two nonintersecting lines on the surface. Their examples include the Clebsch diagonal surface and the Fermat cubic surface **[3]**. Although there is a brief mention that their algorithm could be used to find all of the lines on a surface (save for some minor technicalities), the authors do not expand on this topic. Note that the above algorithm finds equations of lines by solving systems of cubic equations using the Gröbner basis.

5.2 Presentation of original method for finding lines

In this section, we present an original systematic method for finding lines. Since there are finitely many algebraic forms that the parametric equation of a line can take on in projective space, we reduce the problem to several solvable cases. Although our method

is similar to Polo-Blanco & Top's in that it proceeds via cases, our method focuses on the possible algebraic forms a line may assume, rather than searching for factors of the original polynomial. Using this approach, we can find all of the lines on a cubic surface.

We begin with a cubic surface given by $f(x, y, z, w) = 0$ as defined in Section 2.3. Recall that lines are given by parametric homogenous equations $\left(a_0 s + a_1 t,\ b_0 s + b_1 t,\ c_0 s + c_1 t,\ d_0 s + d_1 t\right)$, where not all of $a_1,\ b_1,\ c_1,\ d_1$ are 0. Note that $a_1 = b_1 = c_1 = d_1 = 0$ gives $\left(a_0 s,\ b_0 s,\ c_0 s,\ d_0 s\right) \sim \left(a_0,\ b_0,\ c_0,\ d_0\right)$. Hence, we have an equation for a point in $\mathbf{P}^3(\mathbf{R})$. Also, recall that the "point" (0, 0, 0, 0) does not belong to $\mathbf{P}^3(\mathbf{R})$, so we must exclude it from our solution set.

Step 1: Set $w = 0$.

This equation defines the Zariski-closed subset isomorphic to $\mathbf{P}^2(\mathbf{R})$, contained in $\mathbf{P}^3(\mathbf{R})$, where w is identically 0. We call it the "plane at infinity." Lines in this subset are of the form $\left(a_0 s + a_1 t,\ b_0 s + b_1 t,\ c_0 s + c_1 t,\ 0\right)$, where not all of the coefficients are 0.

We find the lines by examining three subcases—namely, the three Zariski-open sets isomorphic to $\mathbf{R}^2$, contained in the given $\mathbf{P}^2(\mathbf{R})$. These are the sets given by the equations $x = 1$, $y = 1$, and $z = 1$, respectively. Since $\mathbf{R}^2$ is a affine space, the equations need not be homogenous.

Case 1: Let $x = 1$. Lines are of the form $\left(1,\ b_0 + b_1 t,\ c_0 + c_1 t,\ 0\right)$, where one of the $b_1,\ c_1$ must be nonzero.

Case 2: Let $y = 1$. Lines are of the form $\left(a_0 + a_1 t,\ 1,\ c_0 + c_1 t,\ 0\right)$ where one of

the $a_1,\ c_1$ must be nonzero.

Case 3: Let $z = 1$. Lines are of the form $(a_0 + a_1 t,\ b_0 + b_1 t,\ 1,\ 0)$ where one of the $a_1,\ b_1$ must be nonzero.

Note that the equations above correspond to lines in $\mathbf{R}^3$. After solving for the parameters, the equations for the lines can be translated back to homogeneous projective coordinates with $w = 0$. This accounts for all possible lines in the subset given by $w = 0$. Note that all of the sets (planes at infinity) that satisfy the equations $x = 0$, $y = 0$, or $z = 0$ intersect with all the points where $w=0$ in a line. So, the lines that would be contained in the sets given by $x=0$, $y=0$, and $z=0$ are therefore accounted for.

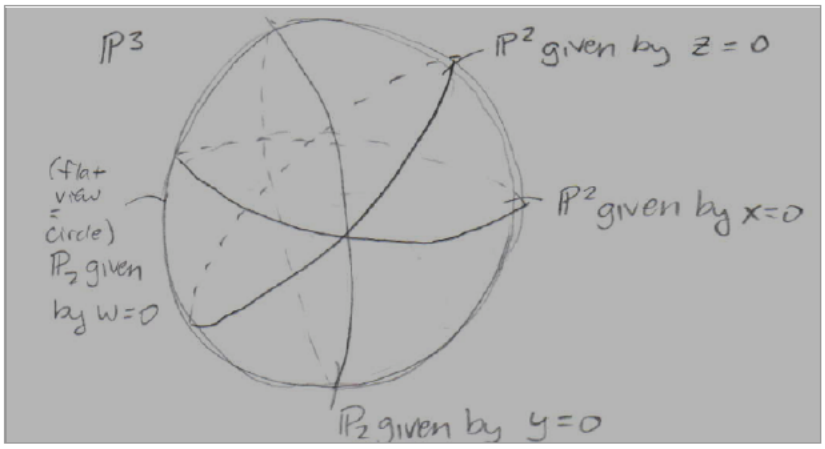


Figure 11. Subsets of $\mathbf{P}^3(\mathbf{R})$.
Figure by author.

Step 2: Set $w = 1$.

This yields the Zariski-open subset of $\mathbf{P}^3(\mathbf{R})$ isomorphic to $\mathbf{R}^3$. Note that the sets $w = 0$ and $w \neq 0$ cover the space $\mathbf{P}^3(\mathbf{R})$. We choose the equation $w = 1$ to describe the set where $w \neq 0$ (recall that all nonzero constants are projectively equivalent, as defined in Section 2.3). Since $\mathbf{R}^3$ is not a homogenous space, we can write equations for lines in form $(a_0 + a_1 t,\ b_0 + b_1 t,\ c_0 + c_1 t)$ such that a_1, b_1, and c_1 are not all 0.

At this point, we need to reduce the number of coefficients in the equation of the line. Currently, there are 6 of them, and our solution method produces at most 4 equations. Note that a linear change of variable will not affect the degree of the polynomial equation describing a line. We know that at least one of a_1, b_1, and c_1 is nonzero, and that the coefficients are the original projective coefficients in the parametric equation of the line. We demonstrate below the convenient change of variable that reduces the number of coefficients.

> Assume that $a_1 \neq 0$. Let T = $a_0 + a_1 t$. Solving for t, we get $t = \frac{T_0 - a_0}{a_1} = \frac{T_0}{a_1} - \frac{a_0}{a_1}$. We introduce this change of variable for t, which gives a line of the form $(T, \beta_0 + \beta_1 T, \gamma_0 + \gamma_1 T)$. Since there were no restrictions on any coefficient other than a_1, we rewrite this line using the original variables, keeping in mind that the constant values may have been affected from our change of variable. This eliminates the coefficients a_0 and a_1, which reduces our system of equations to a four-variable system.

In this way, we have three cases:

> Case 1. If $a_1 \neq 0$, then lines are of the form $(t,\ b_0 + b_1 t,\ c_0 + c_1 t)$.
> Case 2. If $b_1 \neq 0$, then lines are of the form $(a_0 + a_1 t,\ t,\ c_0 + c_1 t)$.
> Case 3. If $c_1 \neq 0$, then lines are of the form $(a_0 + a_1 t,\ b_0 + b_1 t,\ t)$.

Again, after solving for the parameters, the equations for the lines can be translated back to homogeneous projective coordinates with w =1. This accounts for all of the lines contained in the Zariski-open set given by w = 1.

With Step 1 and Step 2 we have covered all possible lines on the surface since the subsets in each step cover $\mathbf{P}^3(\mathbf{R})$. Each of the given cases has at most four unknown constants. In order to solve for these constants, we adopt Polo-Blanco & Top's Maple code. We

demonstrate the steps here, using the Fermat cubic surface given by the equation

$x^3 + y^3 + z^3 + w^3 = 0$ in $\mathbf{P}^3(\mathbf{R})$.

Example 6

1. In Maple, launch the PolynomialTools package.
2. Define the surface.
   F := $x^3 + y^3 + z^3 + w^3$;
2. Create a rule for the form of line for which we are searching. For example, use the form of the line from Step 2, Case 1:
   rule := {x = t; y = $b_0 + b_1 t$; z = $c_0 + c_1 t$; w = 1};
3. We expand the coefficients with
   F_{1a} := expand(subs(rule; F));
4. We collect the coefficients as coefficients of descending powers of *t*.
   F_{2a} := collect(F_{1a}; t);
5. Collect them into a coefficient vector in preparation for solving.
   vect := CoefficientVector(F_{2a}; t);
6. Set the system of equations equal to 0, and solve. Since our unknowns in this case were b0; b1; c0; and c1, we use the four rows of "*vect*" to create a solvable system of four equations for four variables. The command for this is:
   solve({vect[1]; vect[2]; vect[3]; vect[4]}; [b_0; b_1; c_0; c_1];

This yields the coefficients for all real and complex solutions of lines of this form. Note that the solution may be in terms of a free variable if one of the coefficients vanishes in the substitution step. This particular example returns both real and complex solutions, meaning that there are both real and complex lines of this form. Here, we only use the solution sets where all four constants are real numbers. In this example, the two real solution sets of lines on of the form given in Step 2, Case 1 on Fermat's surface are indicated by the parameter values below.

$\{b_0 = -1; b_1 = 0; c_0 = 0; c_1 = -1\}$ and $\{b_0 = 0; b_1 = -1; c_0 = -1; c_1 = 0\}$.

Therefore, the two corresponding lines in $\mathbf{P}^3(\mathbf{R})$ are given parametrically by

$(t,-s,-t,s)$ and $(t,-t,-s,s)$, respectively.

Note that we omit use of the Gröbner basis in this example. The Gröbner basis is used in

this algorithm for reducing the degree of the polynomials in the system of equations in order to solve them. For all of the examples presented in this paper, it was not necessary to use it. However, should there arise a case where the software crashes while trying to solve for the parameters, the Gröbner basis may be used to obtain a solution.

Step 3: Check for duplicates.

In this final step, we consider the set of lines in $\mathbf{P}^3(\mathbf{R})$ that we have found using Steps 1 and 2. The steps and subcases are comprehensive in that they cover all possibilities for lines; however, the subcases are not necessarily mutually exclusive. Therefore, it is necessary to check the set of lines for duplicates. Currently, we are doing this step by hand, but it would be possible to set up a computer program to do so as well.

Recall that any two lines containing the same two points constitute the same line. In this way, we can obtain the set of all distinct lines on the surface.

Example 7

Using the method presented above, we verify that there are three distinct real-valued lines on the Fermat cubic surface given by $x^3 + y^3 + z^3 + w^3 = 0$. We find that these lines are given by (t, -s, -t, s); (t,-t,-s, s); and (-s, t, -t, s). Setting $w = 1$ for visualization, we show them here:

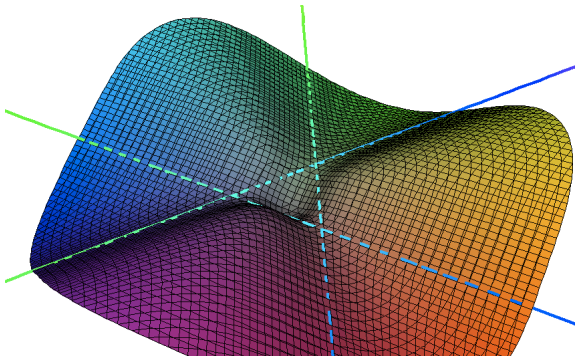


Figure 12. Fermat's lines. Image generated using [12].

Using this method, we find the equations of the lines on the surfaces presented in Section 2.3.2, and list the results in examples 8, 9, and 10.

Example 8

Cayley's nodal surface has 9 lines on it, given parametrically in $\mathbf{P}^3(\mathbf{R})$ by:

$(t, 0, t, 0)$ $(0, t, t, 0)$ $(t, -s, -t, s)$
$(t, 0, 0, t)$ $(0, 0, t, t)$ $(t, -t, -s, s)$
$(t, t, 0, 0)$ $(0, t, 0, t)$ $(-s, t, -t, s)$

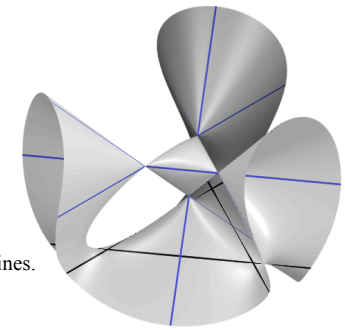


Figure 13. Cayley's nodal surface with lines. Image from [10].

Example 9

The Clebsch diagonal surface has 27 lines on it, given parametrically in $\mathbf{P}^3(\mathbf{R})$ by:

$(s, -s, t, 0)$
$(s, t, -s, 0)$
$(s, t, -t, 0)$
$(t, s, -s, 0)$
$(-s, t, s, 0)$
$(t, -t, s, s)$
$(t, -t, 0, s)$
$(t, -s, -t, s)$
$(t, 0, -t, s)$
$(-s, t, -t, s)$
$(t, \alpha s-t, -s+\alpha t, s)$
$(t, \beta s-t, -s+\beta t, s)$
$(t, \alpha s-t, -\alpha s-\alpha t, s)$
$(t, \beta s-t, -\beta s-\beta t, s)$
$(t, -s+\alpha t, \alpha s-t, s)$
$(t, -s+\beta t, \beta s-t, s)$
$(t, -\alpha s-\alpha t, \alpha s-t, s)$
$(t, -\beta s-\beta t, \beta s-t, s)$
$(t, -s-\alpha t, \alpha s+\alpha t, s)$
$(t, -s-\beta t, \beta s+\beta t, s)$
$(t, \alpha s+\alpha t, -s-\alpha t, s)$
$(t, \beta s+\beta t, -s-\beta t, s)$

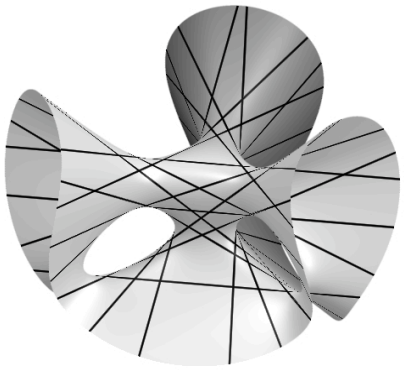


Figure 14. Clebsch diagonal surface with lines. Image from [10].

where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$.

Example 10

The ding-dong surface has no lines lying on it.

5.3 Symmetry of a surface

Another important geometric descriptor of a surface in $\mathbf{P}^3(\mathbf{R})$ is its symmetry group. The symmetries of a surface are determined by investigating when that surface is invariant under a rigid-motion transformation in $\mathbf{P}^3(\mathbf{R})$, such as rotations about an axis and/or reflections across a plane. The symmetries of any surface form an algebraic group of transformations under composition.

The set of all isometric transformations in $\mathbf{P}^3(\mathbf{R})$ is can be represented as a matrix group PSL(3,**R**), the projective special linear group of order 3. In other words, $PSL(3,R) \cong {}^{SL(4,R)}\!/_{\sim}$, where SL(4,**R**) is the special linear group of order 4, and ~ is the equivalence relation from Definition 1. Because of this, the group of symmetries exhibited by any surface in $\mathbf{P}^3(\mathbf{R})$ is necessarily a subgroup of PSL(3,**R**). These groups may have infinite order. By analyzing the symmetry of a surface, we can identify members of these subgroups that may indicate the overall symmetric structure.

5.4 Presentation of original application of level curves

We graph the level curves of a surface in order to analyze the rotational and reflective symmetry about an axis or in a plane, respectively.

Example 11

Consider the following level-curves analysis of Fermat's cubic surface. After visualizing the affine part of the surface in $\mathbf{R}^3$, given by $x^3 + y^3 + z^3 + 1 = 0$, we graph some level curves of the surface by replacing each coordinate by a series of constants. This yields three sets of level curves (one for each canonical axis).

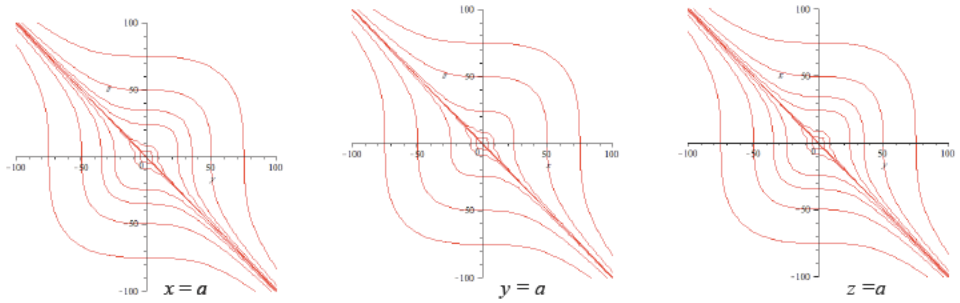


Figure 15. Fermat level curves. Image generated using [14].

Figure 15 shows the level curves of Fermat's cubic surface for each canonical axis. However, it should be noted that any plane may be used to generate level curves that can then be used to analyze symmetries outside of the canonical axes.

From the level curves shown when $x = a$, it is clear that the surface exhibits the symmetry of rotations by π about the x-axis (r_{180}^{x}), as well as reflections in the plane given by $y + z = 0$ (m_1). Note that the level curves for $y = a$ and $z = a$ are identical to those for $x = a$. Therefore, the surface also exhibits π-rotational symmetry about the y- and z-axes (denoted r_{180}^{y} and r_{180}^{z}, respectively) and reflections across the planes given by $x + z = 0$ and $x + y = 0$ (denoted m_2 and m_3). So the group includes $\{r_{180}^{x}, r_{180}^{y}, r_{180}^{z}, m_1, m_2, m_3, I\}$.

Example 12

Cayley's nodal surface

Affine equation: $4(x^2 + y^2 + z^2) + 16xyz - 1 = 0$ **[14]**

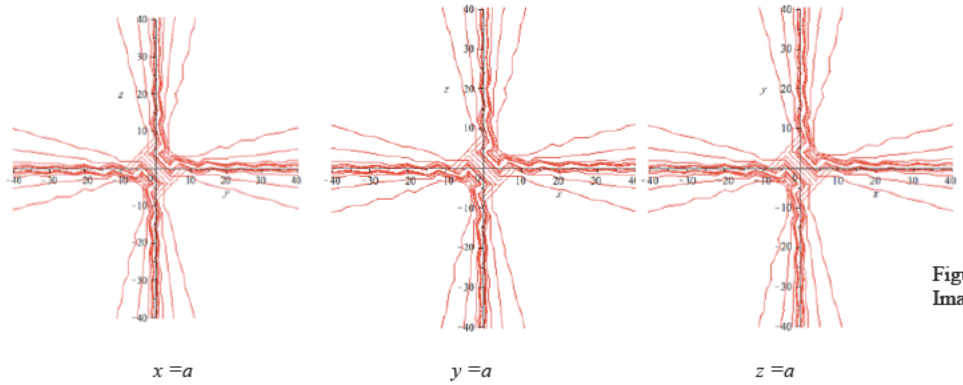


Figure 16. Cayley level curves. Image generated using [14].

Symmetry: $\{r_{180}^{x}, r_{180}^{y}, r_{180}^{z}, m_{1x}, m_{2x}, m_{1y}, m_{2y}, m_{1z}, m_{2z}, I\}$

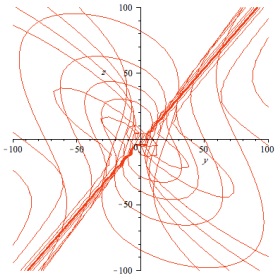
where m_{1x} is a reflection across the plane given by $z+y = 0$;
m_{2x} is a reflection across the plane given by $z-y = 0$;
m_{1y} is a reflection across the plane given by $x+z = 0$;
m_{2y} is a reflection across the plane given by $x-z = 0$;
m_{1z} is a reflection across the plane given by $x+y = 0$;
m_{2z} is a reflection across the plane given by $x-y = 0$;

Example 13

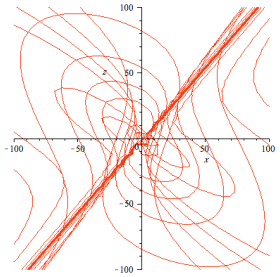
Clebsch diagonal surface

Affine equation:

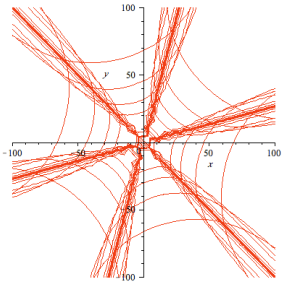
$$16x^3 + 16y^3 - 31z^3 + 24x^2z - 48x^2y - 48xy^2 + 24y^2z - 54\sqrt{3}z^2 - 72z = 0 \quad \textbf{[14]}$$



$x = a$



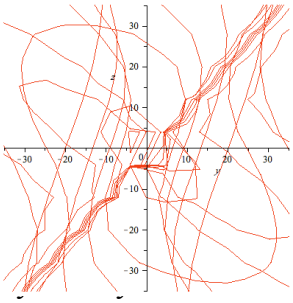
$y = a$

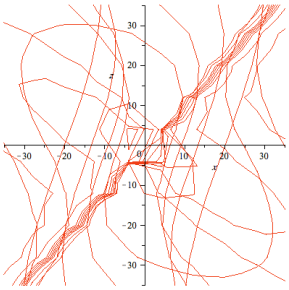


$z = a$

Figure 17. Clebsch level curves. Image generated using [14].

Zoomed in on center:





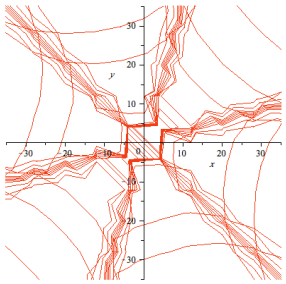


Figure 18. Zoomed Clebsch level curves. Image generated using [14].

The symmetry exhibited by this surface is interesting because, on a large scale, there appear to be more symmetries present. However, the behavior of these level curves by the origin "breaks" most of these, as is evident from Figure 18.

While this information gives valuable insight into the group structure of the surface, the symmetries found using this method are not comprehensive. There may be other

symmetries present in this surface that the method has not detected. For example, the symmetry group for the Clebsch diagonal surface is the subgroup of orientation-preserving transformations of S_5 **[1]**. However, we have constructed a set of symmetries present in the surface that may be indicative of its overall structure. We can also list specific elements of the group. The value of this method lies in the fact that it allows for a visual analysis of symmetry for any surface, as well as the fact that it may be applied to any algebraic surface.

An interesting consequence of this method is that it can also be used to identify curves on a surface. In Example 7, we found that there are three real-valued lines on the Fermat cubic surface, given by (*t, -s, t, s*)*;* (*t, -t, -s, s*)*;* and (*-s, t, -t, s*). In Figure 15, when $a = 0$, the plane $x=a$ yields a level curve for the surface given by the line $y = -z$. This corresponds exactly to the known line on the surface given in $\mathbf{P}^3(\mathbf{R})$ by (*-s*, *t*, *-t*, *-s*); notice that the *y* and *z* coordinate values of this line are of the same magnitude and opposite sign.

In general, level curves may be used to identify any curve on an algebraic surface. For example, the "ding-dong" cubic surface has no lines that lie on it, but it does have infinitely many circles. This fact can be easily verified using level curves.

Example 14

The ding-dong surface is given in $\mathbf{P}^3(\mathbf{R})$ by $z^3 + x^2w + y^2w - z^2w = 0$. Letting $w = 1$ for visualization, we can easily see that the level curves in planes given by $z = a$ are concentric circles.

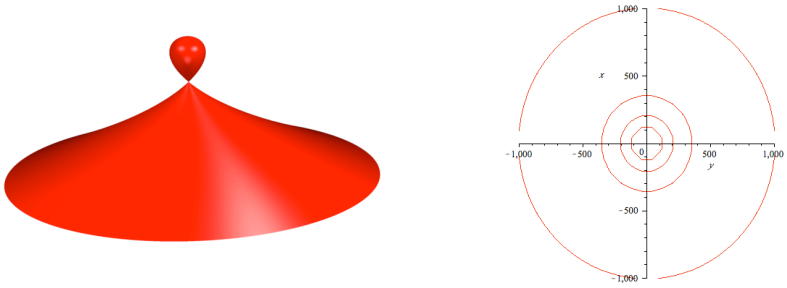


Figure 19. Ding-dong level curves. Image generated using [4] and [11].

5.5 Singularities

In this section, we review the use of Jacobi's theorem to identify singularities on cubic surfaces. Historically, many cubic surface classifications rely heavily on the presence and type of singularities. Bruce & Wall's comprehensive classification of nodes, Knörrer and Miller's toplogical classification, and Holzer & Labs' gallery of cubic surfaces provide an excellent library of information regarding this geometric feature **[2,8,11]**. Because of this, we have chosen to focus more on other geometric properties of surfaces. However, we are interested in identifying, and finding the exact coordinates for, singular points on a cubic surface. To do so, we apply Jacobi's theorem **[21]**.

> Theorem (Jacobi): Let S be a surface given by a single homogenous polynomial equation $f(x,y,z,w) = 0$. A point c that lies on S is a singularity if and only if $f_x(c) = f_y(c) = f_z(c) = 0$.

We can easily apply this theorem by considering two cases. We first allow $w = 1$ to look for singular points in $\mathbf{R}^3$. We then take the implicit partial derivatives with respect to x, y, and z of $f(x, y, z, 1)$. We set these three equations equal to 0. Any solution to the system that also lies on the surface corresponds to a singular point. We repeat the procedure for

w = 0 to find possible singular points at infinity. We demonstrate this method on Fermat's cubic surface in Example 15.

Example 15

Step 1: Let w =1. The affine surface is given by $x^3 + y^3 + z^3 + 1 = 0$.

Taking partial derivatives, $\frac{\partial f}{\partial x} = 3x^2$, $\frac{\partial f}{\partial y} = 3y^2$, and $\frac{\partial f}{\partial z} = 3z^2$. We set each equation to 0, yielding the system $3x^2 = 0,\ 3y^2 = 0,\ 3z^2 = 0$. The only solution is (0,0,0). However, the point (0,0,0,1) does not lie on Fermat's cubic surface. So there are no singular points found in this step.

Step 2: Let $w = 0$. The surface is given in $\mathbf{P}^2(\mathbf{R})$ by $x^3 + y^3 + z^3 = 0$. As in our method for finding lines, we check the three subsets of $\mathbf{P}^2(\mathbf{R})$ isomorphic to $\mathbf{R}^2$ given by x =1, y = 1, and z = 1, respectively. Carrying out a similar analysis of partial derivatives shows that the only solutions to these systems are (1,0,0,0); (0,1,0,0); and (0,0,1,0), none of which are points on Fermat's surface.

Recall from section 5.2 that these two cases cover all of $\mathbf{P}^3(\mathbf{R})$. Therefore, we can conclude that there are no singular points on this surface. In other words, the surface is smooth (see definition in Appendix I).

Example 16

Cayley's nodal surface

There are 4 singular points on Cayley's nodal surface. Let the surface be given in $\mathbf{P}^3(\mathbf{R})$ by $xyz + xyw + xzw + yzw = 0$. Then, the singular points are as follows:

(1,0,0,0) (0,0,1,0)
(0,0,0,1) (0,1,0,0)

Example 17

Clebsch diagonal surface

There are no singular points on the Clebsch diagonal surface. In other words, it is smooth.

Example 18

Ding-dong surface

There are two singular points on the ding-dong surface, located at (1,1,0,1) and (0,0,0,1). Note that the second singular point is usually the only one visible in graphs of the affine equation.

5.6 Additional Geometric Properties

Although our study focused on the three geometric properties demonstrated above, there are many others that are immediately obvious upon visualization of the surface in $\mathbf{R}^3$.

- The number of "holes" it has. This is related to the genus of the surface.
- Connected: An algebraic surface is *connected* if, between any two points, there is a continuous path connecting them.
- Number of components: The number of *disconnected* components, or pieces.
- Bounded: A surface is *bounded* if there exists a ball of finite radius that completely contains the entire surface. In $\mathbf{R}^3$, it is by definition impossible for any surface of odd degree to be bounded.

Note that some of these, such as genus and connectedness, are topological descriptors for a surface. These properties are often important for describing higher-degree algebraic surfaces that do not necessarily exhibit consistent geometric features (such as lines).

6 Surface Deformations

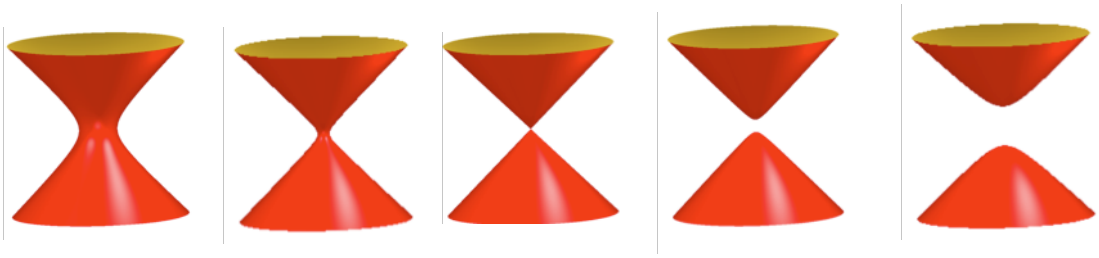
In this section, we explore deformations of cubic surfaces. Although any linear transformation constitutes a deformation of a cubic surface, we consider the simplest deformation available: varying the original equation of the surface in a one-parameter family.

6.1 Geometric Properties and Deformation

Ideally, properties used for classification would be invariant under a simple deformation or small perturbation. However, we find that this is not necessarily the case. Even under the simple deformation given by varying a single constant, the geometric properties of a surface are not necessarily preserved. This phenomenon is not unique to cubic surfaces; consider the following deformation from a hyperboloid of one sheet, to a cone, to a hyperboloid of two sheets.

Example 19

$$x^2 + y^2 - z^2 = a$$



$a = 0.5$ $a = 0.05$ $a = 0$ $a = -0.05$ $a = -0.5$

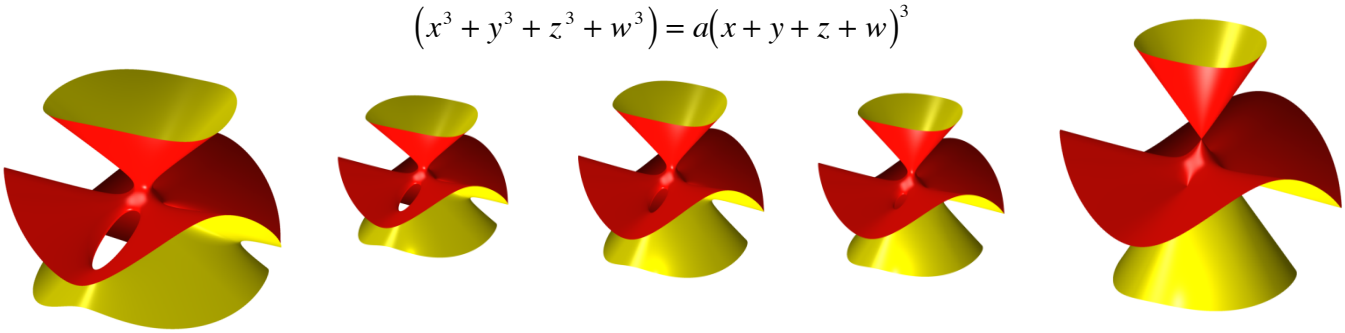
Figure 20. Quadratic deformation. Images generated using [4].

Note that in this example, the geometric properties of the surface are not preserved—the surface goes from smooth to singular, and back again; and the lines on the surface disappear as we go from a double-ruled surface, to a ruled surface, to a surface with no

lines on it. However, the symmetry, and therefore the algebraic group structure, *is* preserved in this deformation. It is also possible to deform several geometric properties at once, as seen in Example 20.

Example 20
Consider the following smooth deformation from the Clebsch diagonal surface to Cayley's nodal surface. The following visualization is given with $w = 1$.

$$\left(x^3 + y^3 + z^3 + w^3\right) = a\left(x + y + z + w\right)^3$$



Clebsch diagonal surface
$a = 1$

Cayley's nodal surface
$a = 0.5$

Figure 21. Clebsch – Cayley deformation.
Images generated using [4].

Both of these surfaces exhibit such interesting geometric properties that Schlafli constructed a special class for each of them in his 1858 paper. Notably, the Clebsch diagonal surface is the only smooth cubic surface for which all 27 of its lines are real-valued; while Cayley's nodal surface is the only cubic surface with 4 nodal points (each of these unique up to projective isomorphism). Note that this particular visualization of these surfaces appear different from others in the text; however, the other visualizations simply correspond to different choices of w in the homogenous projective equation.

6.2 Deformation Families

A deformation family is a series of smooth deformations (in this case, by a single constant) designed to either preserve or affect certain geometric properties of the surface.

Example 21

In this example, we present a deformation family for Cayley's nodal surface that preserves its symmetry, but affects its singularities and lines. In this example, we use a particular affine representation of the surface that clearly shows its symmetries and singularities, given by **[7]**.

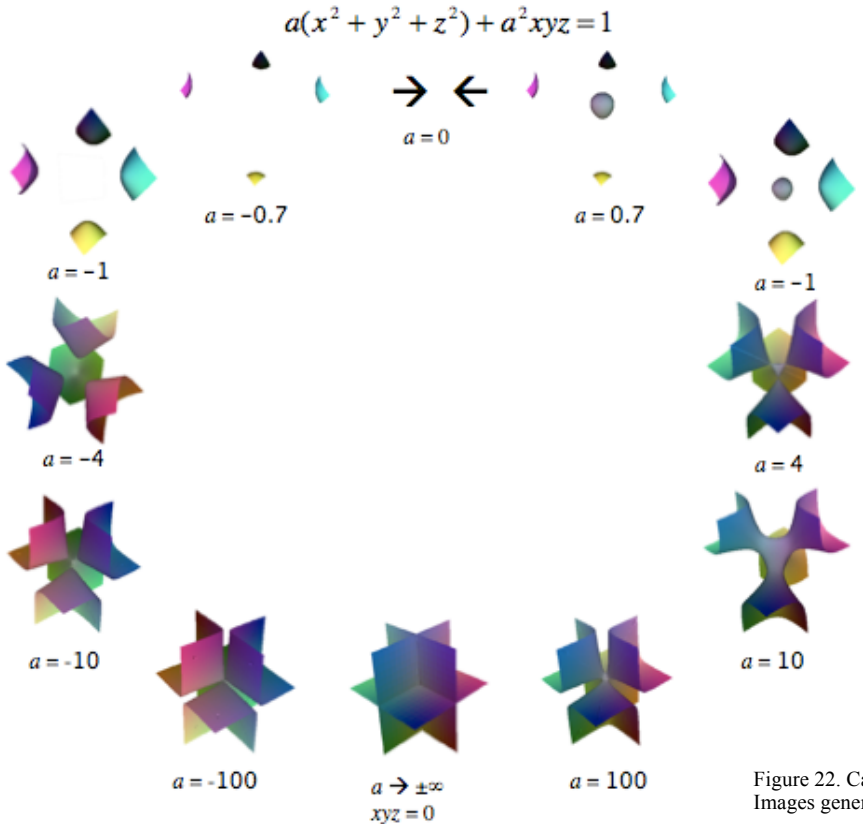


Figure 22. Cayley deformation family. Images generated using [15].

Effect on Nodal Points

S is smooth for any real number $a \neq 4$.

Effect on Lines

When $a = 4$, S has 6 lines.
When $a > 4$, S has 24 lines.
When $a < 4$, S has 0 lines.

Effect on Symmetry

Symmetry of the surface is preserved in deformation.

Degeneration

S approaches the degenerate surface $xyz = 0$ as $a \to \pm\infty$.
S degenerates into a no-solution scenario as $a \to 0$.

For more examples of deformation families, the reader should reference R. Morris' SingSurf website **[15]** and Holzer & Labs' gallery of cubic surfaces **[10]**. Morris provides several excellent examples of simple deformations that result in different types of singularities. Holzer & Labs include dozens of animated smooth deformation movies, showing how many of the 45 surfaces with rational double points can be deformed into one another.

The idea of deformation families of cubic surfaces introduces a "metric" of sorts on a classification of surfaces. Given two surfaces that are classified differently due to the differences in their geometric properties, they could still be considered "close" in some sense if there exists a smooth, continuous deformation from one surface to the other. This concept adds another level of depth to a meaningful classification for cubic surfaces. Experimenting with deformation families for cubic surfaces contributed to this research in that we were able to study trends in how geometric properties of a surface vary with even a simple deformation.

7 Summary

In this paper we presented an original method for finding all of the real-valued lines on any cubic surface, as well as determining the explicit parametric equations for these lines. This method is based on the fact that there are finitely many forms possible for real-valued lines in projective space, and that these forms can be reduced to several solvable systems of equations. In addition to this, the application of level curves to cubic surfaces in $\mathbf{R}^3$ allows an intuitive visual analysis of the symmetry of the surface.

This method builds upon the known geometric properties of cubic surfaces in $\mathbf{P}^3(\mathbf{R})$ in the context of the larger unsolved problem of geometric classification of cubic surfaces. First, we reviewed the history of the problem and the progress towards a solution, keeping in mind in the goals of a "good" classification. We then visualized real surfaces in this space via an affine equation and studied several intrinsic geometric properties of the surfaces both visually and analytically. In particular, we analyzed the presence and explicit parametric equations of real valued lines on the surface, the group of symmetries exhibited by the surface, and the presence and location of singular points. We reviewed and tested methods for analyzing these properties and presented some new ideas as well. Finally, we studied deformations of cubic surfaces; in particular, forming deformation families to either preserve or affect geometric properties, as well as exploring the possibility of a continuous deformation from one surface to another. We found that even a simple deformation may affect the geometric properties of the original surface, and even change properties that are considered "defining" characteristics.

Our major contribution to the open problem is the methods presented for meaningful analysis of important geometric properties of cubic surfaces. The reader will note that the examples given in the text are for well-known surfaces. The purpose of analyzing these surfaces was to test the accuracy and robustness of the methods, since many of their properties are already well-known. Directions for further research include the application of these methods to *any* cubic surface, regardless of whether it has been previously analyzed. We are confident that these methods can be applied to any cubic surface.

To further this goal, a group of undergraduate students from CSU Channel Islands have been exploring the geometric properties of several of the classes of surfaces given in Schlafli's classification. These students have applied the methods presented in this paper in order to meaningfully and explicitly describe the geometric properties of these surfaces.

Some additional applications of this work for further research include: further study of other algebraic curves lying on the surface using analysis of level curves; writing computer programs to automatically carry out the analysis for the lines and symmetry; and further construction of deformation families using randomly generated cubic surfaces. There are also numerous "smaller" unsolved problems related to geometric classification; for example, constructing equations for cubic surfaces with a requested set of properties or proving that none exists.

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APPENDIX I
Analysis of Geometric Properties of Quadratic Surfaces

A quadratic surface in $\mathbf{P}^3(\mathbf{R})$ is given by a single degree two homogenous polynomial equation

$$f(x,y,z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xz + a_5xy + a_6yz + a_7xw + a_8yw + a_9zw + a_{10}w^2 = 0$$

where a_i is any real number, not all 0.

To visualize these surfaces in $\mathbf{R}^3$, we set $w = 1$. In $\mathbf{R}^3$, the most general equation for degree two algebraic surfaces is as follows:

$$Ax^2 + By^2 + Cz^2 + Dxz + Exy + Fyz + Gx + Hy + Iz + J = 0$$

where each constant is a real number, and at least one of *A, B, C, D, E,* or *F* is nonzero.

By translation and rotation, this equation can be brought into one of two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0$$

or

$$Ax^2 + By^2 + Iz = 0$$

There are 17 types of degree 2 surfaces. They are:

1. Circular cone
2. Elliptic cone
3. Circular cylinder
4. Elliptic cylinder
5. Ellipsoid
6. Spheroid
7. Sphere
8. Hyperboloid of 1 sheet
9. Hyperboloid of 2 sheets
10. Elliptic paraboloid
11. Paraboloid of revolution
12. Hyperbolic cylinder
13. Parabolic cylinder
14. Hyperbolic paraboloid

Degenerate cases:

15. Intersecting planes
16. Double plane
17. Parallel plane

There are also cases where a quadratic equation may have exactly one, or no, real solutions. Because these solution sets do not form an algebraic surface, they are not included in the classification of quadratic surfaces.

We will analyze the geometry of each of these surfaces by studying their canonical-form equations in $\mathbf{R}^3$, as well as several geometric properties. These include boundedness, symmetries (reflections and rotations), smoothness, number of pieces, possible singularities, maximums, minimums, saddle points, ruled, and self-intersection.

Note:
Any of these surfaces can be rotated and/or translated to move them around in R^3. However, for simplicity's sake, we will assume that each of these surfaces is centered at the origin. In order to translate the surfaces, one would need to introduce a shift into the canonical-form equation. In order to rotate the surfaces, one would need to employ a transformation matrix. Because our surfaces are centered at the origin, we will discuss symmetries in terms of the *x*-, *y*-, and *z*-axes.

Definitions:
Bounded: A surface is bounded if there exists a ball of finite radius containing it.

Reflective symmetries: obtained by reflecting a surface across a plane. We can analyze these symmetries visually, as well as verify them algebraically. For example, consider a surface that appears to have reflective symmetry across the *xy* plane. If this is true, then $f(x, y, z, 1) = f(x, y, -z, 1)$.

Rotational symmetries: obtained by rotating the surface by some degree around a given axis. These can be algebraically verified by applying a 3x3 rotation matrix to the equation of the surface in $\mathbf{R}^3$.

Smoothness: A surface is smooth if it is differentiable at any point.

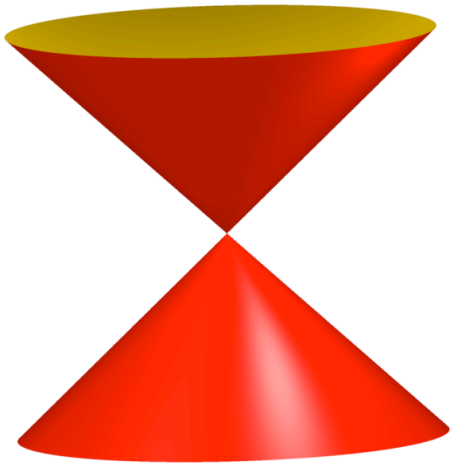
Singularity: A singularity is a point on a surface at which the tangent plane to the surface is undefined.

Ruled: A surface is ruled if , through any point on the surface, there exists a straight line on the surface passing through that point. A double-ruled surface has two distinct lines for each point.

π-measurable rotation: A rotation, such that if it is repeated a finite number of times, results in the surface rotating by some integer multiple of π.

Circular Cone

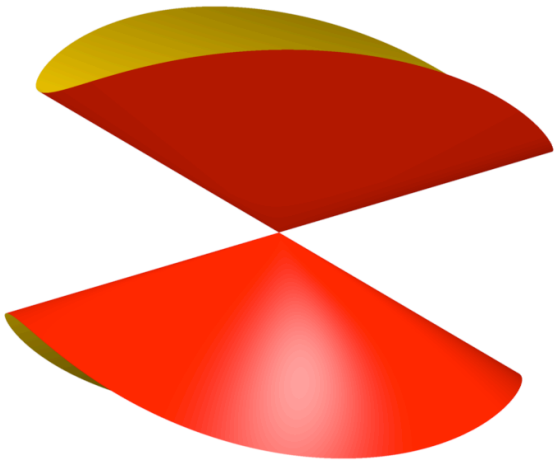
Standard Form Equation	$x^2 + y^2 = az^2$ As *a* gets larger (>1) the cone becomes skinny. As *a* gets smaller (<1), the cone widens.
Reflective Symmetry	Symmetric over *xy* plane Symmetric over any plane that intersects the *z*-axis in a line
Rotational Symmetry	*x*-axis: π, 2π (identity) *y*-axis: π, 2π *z*-axis: Symmetric about all π-measurable rotations of the *z*-axis
Bounded	No
Smoothness	Smooth everywhere except singularity
Number Of pieces	1
Self-intersections	Not possible
Singularities	One singularity at (0,0,0).
Maximums	None
Minimums	None
Saddle points	None
Ruled	Yes



Elliptic Cone

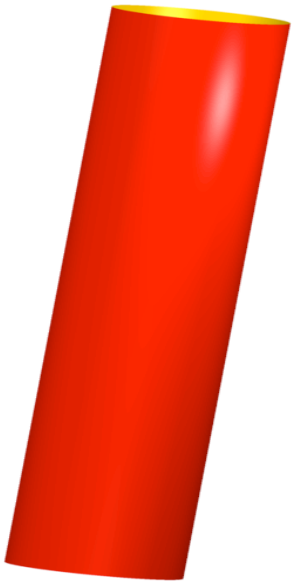
This cone is very similar to the previous one. However, the cross-sectional slices in planes parallel to the *xy* plane are ellipses, not circles. This restricts the symmetry of the elliptic cone as compared to the circular cone.

Standard Form Equation	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$ As *c* gets larger (> 1), the cone becomes skinnier. As *c* gets smaller (< 1), the cone widens. *a* and *b* control the width of the cross-sectional ellipses in the x and y directions, respectively.
Reflective Symmetry	Symmetric over *xy*, *xz*, *yz* planes
Rotational Symmetry	*x*-axis: π, 2π *y*-axis: π, 2π *z*-axis: π, 2π
Bounded	No
Smoothness	Smooth everywhere except singularity
Number Of pieces	1
Self-intersections	Not possible
Singularities	One singularity at the origin.
Maximums	None
Minimums	None
Saddle points	None
Ruled	Yes



Circular Cylinder

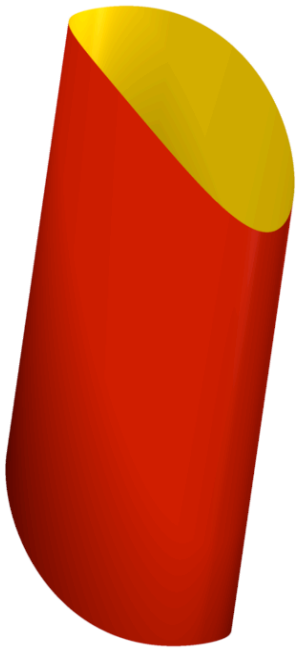
Standard Form Equation	$x^2 + y^2 = a^2$ Note that the z-coordinate is missing; in $\mathbf{R}^3$ this means that the circle in the xy plane extends infinitely in the z-direction. Constant a controls the radius of the circle in the xy plane.
Reflective Symmetry	Symmetric over xy plane Symmetric over any plane that intersects the z-axis in a line
Rotational Symmetry	x-axis: π, 2π y-axis: π, 2π z-axis: Symmetric about all π-measurable rotations of the z-axis
Bounded	No
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	None
Minimums	None
Saddle points	None
Ruled	Yes



Elliptic Cylinder

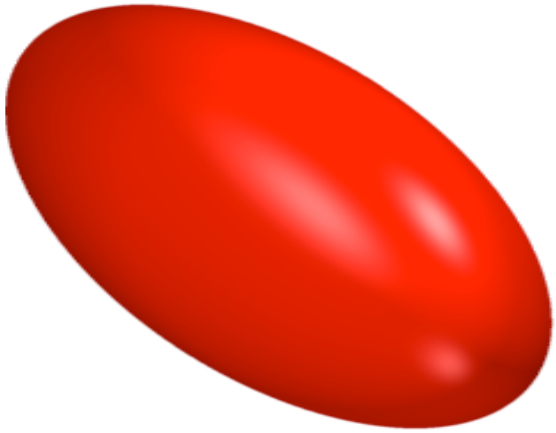
This cylinder is very similar to the previous one. However, the cross-sectional slices in planes parallel to the *xy* plane are ellipses, not circles. This restricts the symmetry of the elliptic cylinder as compared to the circular cylinder.

Standard Form Equation	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$ Note that the *z*-coordinate is missing; in $\mathbf{R}^3$ this means that the circle in the *xy* plane extends infinitely in the *z*-direction. Constants *a, b* control the ellipse in the *xy* plane.
Reflective Symmetry	Symmetric over *xy, xz, yz* planes
Rotational Symmetry	*x*-axis: π, 2π *y*-axis: π, 2π *z*-axis: π, 2π
Bounded	No
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	None
Minimums	None
Saddle points	None
Ruled	Yes



Ellipsoid

Standard Form Equation	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ *a, b, c* distinct real numbers, corresponding to the radii in the *x*, *y* and *z* directions, respectively.
Reflective Symmetry	Symmetric over *xy, xz, yz* planes.
Rotational Symmetry	*x*-axis: 2π *y*-axis: 2π *z*-axis: π, 2π
Bounded	Yes
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	Maximums will always be found on the three axes of rotational symmetry. In this orientation: On *x*-axis: (*a*, 0, 0) On *y*-axis: (0, *b*, 0) On *z*-axis: (0, 0, *c*)
Minimums	Minimums will always be found on the three axes of rotational symmetry. In this orientation: On *x*-axis: (-*a*, 0, 0) On *y*-axis: (0, -*b*, 0) On *z*-axis: (0, 0, -*c*)
Saddle points	None
Ruled	Not ruled



Spheroid

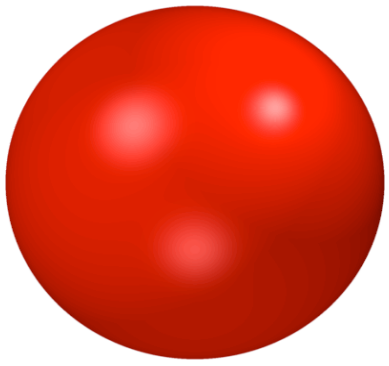
This is a specific type of ellipsoid where two of the constants in the standard-form equation are equivalent.

Standard Form Equation	$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1$ where *a, b* are distinct real numbers corresponding to the radii in the *x* and *y* directions, and the *z* direction, respectively.
Reflective Symmetry	Symmetric over *xy* plane. Symmetric over any plane intersecting the *z*-axis in a line.
Rotational Symmetry	*x*-axis: 2π *y*-axis: 2π *z*-axis: Radially symmetric about any π -measurable rotation
Bounded	Yes
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	Maximums will always be found on the three axes of rotational symmetry. In this orientation: On *x*-axis: (*a*,0,0) On *y*-axis: (0,*a*,0) On *z*-axis: (0,0,*b*)
Minimums	Minimums will always be found on the three axes of rotational symmetry. In this orientation: On *x*-axis: (-*a*,0,0) On *y*-axis: (0,-*a*,0) On *z*-axis: (0,0,-*b*)
Saddle points	None
Ruled	Not ruled

Sphere

This is the most restricted type of ellipsoid, where all of the constants in the standard form equation are equal.

Standard Form Equation	$x^2 + y^2 + z^2 = a^2$ where a is the radius of the sphere.
Reflective Symmetry	Symmetric through any plane that intersects the origin
Rotational Symmetry	Symmetric about any π-measurable rotation about any axis
Bounded	Yes
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	Maximums will always be found on the three axes of rotational symmetry. In this case: On x-axis: $(a, 0, 0)$ On y-axis: $(0, a, 0)$ On z-axis: $(0, 0, a)$
Minimums	Maximums will always be found on the three axes of rotational symmetry. In this case: On x-axis: $(-a, 0, 0)$ On y-axis: $(0, -a, 0)$ On z-axis: $(0, 0, -a)$
Saddle points	None
Ruled	Not ruled

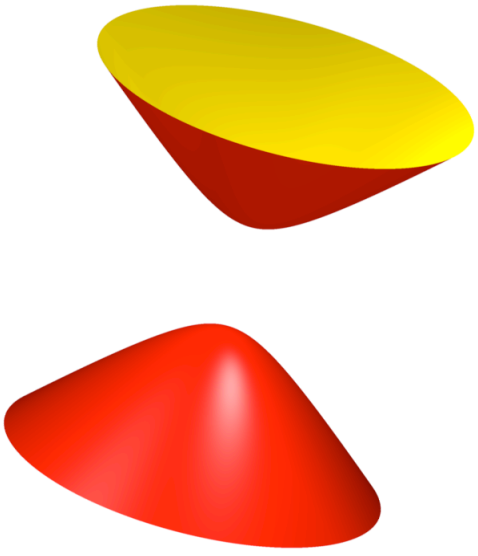


Hyperboloid of One Sheet

Standard Form Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ The hyperboloid will be oriented in the direction of the negative variable.
Reflective Symmetry	Symmetric over *xy, xz, yz* planes
Rotational Symmetry	*x*-axis: π, 2π *y*-axis: π, 2π *z*-axis: π, 2π
Bounded	No
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	None
Minimums	None
Saddle points	None
Ruled	Double Ruled

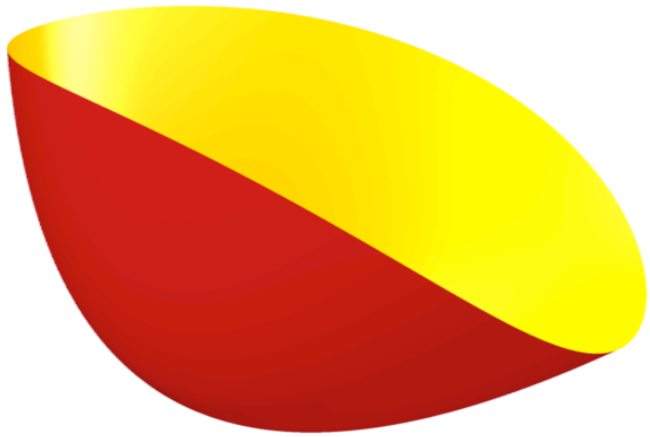
Hyperboloid of Two Sheets

Standard Form Equation	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ The hyperboloid will be oriented in the direction of the positive variable.
Reflective Symmetry	Symmetric over *xy, xz, yz* planes
Rotational Symmetry	x-axis: π, 2π y-axis: π, 2π z-axis: π, 2π
Bounded	No
Smoothness	Smooth everywhere
Number Of pieces	2
Self-intersections	Not possible
Singularities	None
Maximums	In the bottom piece, there is a maximum *z*-value, which depends on the constant *c*
Minimums	In the top piece, there is a minimum *z*-value, which depends on the constant *c*
Saddle points	None
Ruled	Not ruled



Elliptic Paraboloid

Standard Form Equation	$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=cz$$ The elliptic paraboloid will be oriented in the direction of the linear term (taking its sign into account).
Reflective Symmetry	Symmetric over *xy, xz, yz* planes
Rotational Symmetry	x-axis: 2π y-axis: 2π z-axis: π, 2π
Bounded	No
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	None
Minimums	There is a minimum *z*-value at the origin.
Saddle points	None
Ruled	Not ruled



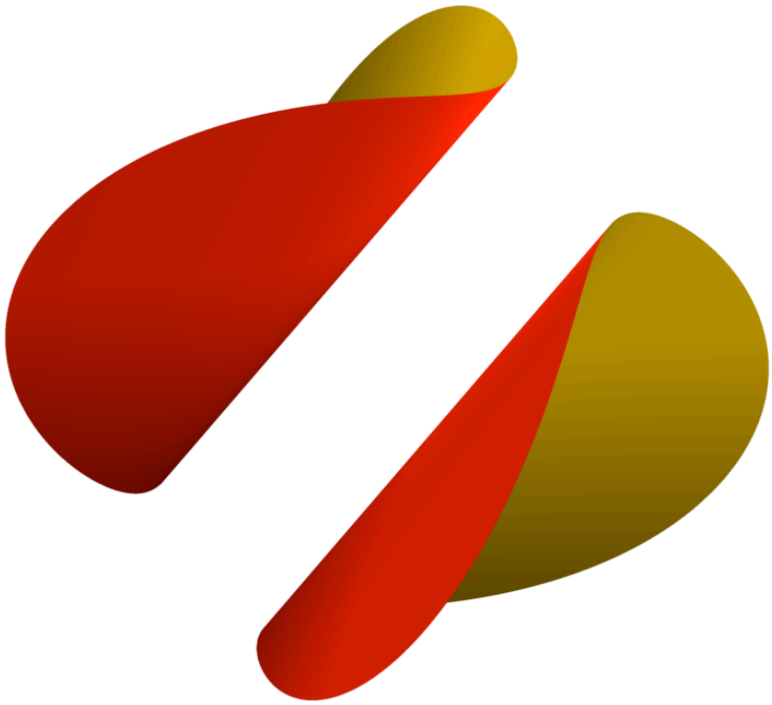
Paraboloid of Revolution

This is similar to the elliptic paraboloid, but any cross-section parallel to the *xy* plane is a circle instead of an ellipse.

Standard Form Equation	$x^2 + y^2 = az$ The paraboloid will be oriented in the direction of the linear (taking its sign into account).
Reflective Symmetry	Symmetric over any plane that intersects the *z*-axis in a line
Rotational Symmetry	*x*-axis: 2π *y*-axis: 2π *z*-axis: Any π-measurable degree of rotation (radially symmetric about *z*-axis).
Bounded	No
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	None
Minimums	There is a minimum *z*-value at the origin.
Saddle points	None
Ruled	Not ruled

Hyperbolic Cylinder

Standard Form Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Since the z-coordinate is missing, the surface extends infinitely that direction. Also, because the x-term has positive sign, the surface "opens up" over the x-axis.
Reflective Symmetry	Symmetric over *xy, yz, xz* planes
Rotational Symmetry	x-axis: π, 2π y-axis: π, 2π z-axis: π, 2π
Bounded	No
Smoothness	Smooth everywhere
Number Of pieces	2
Self-intersections	Not possible
Singularities	None
Maximums	None
Minimums	On each piece, there is a minimum x-value; it depends on the constant a. As $\|a\|$ decreases, the two pieces get closer and closer together (i.e., the minimum x-value on each piece gets closer and closer to 0).
Saddle points	None
Ruled	Yes

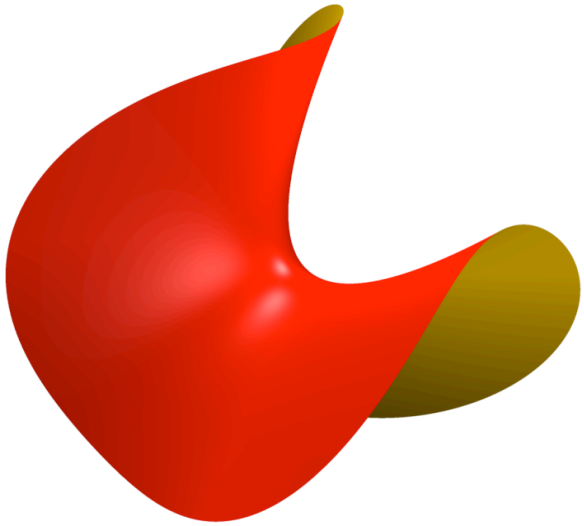


Parabolic Cylinder

Standard Form Equation	$x^2 + 2az = 0$ Since the *y*-coordinate is missing, the surface extends infinitely that direction.
Reflective Symmetry	Symmetric over *xy, yz* planes
Rotational Symmetry	*x*-axis: π, 2π *y*-axis: 2π *z*-axis: 2π
Bounded	No
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	None
Minimums	Minimum along line given by (0, *t*, 0)
Saddle points	None
Ruled	Yes

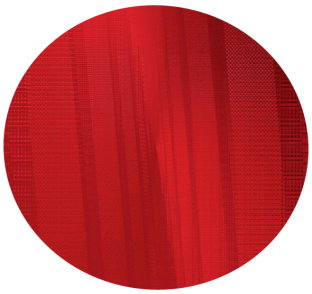
Hyperbolic Paraboloid

Standard Form Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
Reflective Symmetry	Symmetric over *xy, yz, xz* planes
Rotational Symmetry	*x*-axis: 2π *y*-axis: 2π *z*-axis: π, 2π
Bounded	No
Smoothness	Smooth everywhere
Number Of pieces	1
Self-intersections	Not possible
Singularities	None
Maximums	None
Minimums	None
Saddle points	One at the origin
Ruled	Yes



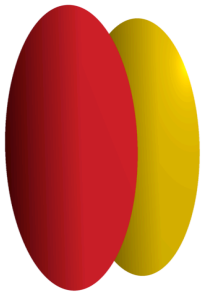
Degenerate Cases

Double Plane



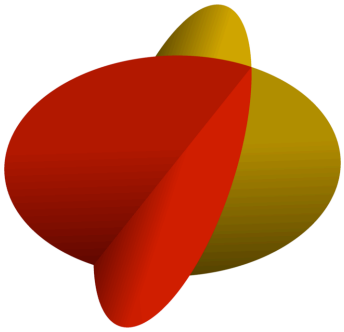
Example: $x^2 = 1$

Parallel Planes



Example: $(x-1)(x+1) = 0$

Intersecting Planes



Example: $x^2 - y^2 = 0$