Smoothing the Transition to Higher Level Mathematics with Technology

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CHAPTER 1

Introduction:

Functions are one of the first abstract concepts students are faced with in mathematics and this topic typically is used as an introduction to higher mathematical thought. In higher level mathematics courses, students are asked to think about their answers and how to approach the problem before they begin to answer the questions. They are no longer able to simply mirror basic examples by changing the numbers to match the example in the book. They must also be prepared to explain their answers and extrapolate information that is not always clear. In the case of functions, they have to think about what it means to be a function and then apply that to given scenarios. This is where we as educators tend to lose many students' interests and start to see a division between students who understand, and those that don't. Why is this? Why are functions so hard to teach? And finally, how can we smooth this transition to higher level mathematics? These are the questions we will study in this paper.

Technology has changed dramatically in the last 20+ years, but teaching styles have remained relatively the same. Many mathematics teachers use a large text book and attempt to teach the concept by doing a few examples on the board and then assign homework. The biggest difference I have seen since I was in grade school is the change of the color of the board used in the lecture. It used to be green and teachers used chalk to write on it, and now we use a white board with erasable markers. Still this 'technology' does not take advantage of the way students communicate today. They are using computers or texting on cell phones to share their ideas. This is the medium that we need to take advantage of.

I set out to find a way to smooth this transition to higher level mathematics by studying the influence of a PowerPoint lecture on functions that contained plenty of animations and a hyperlink to an applet [8] movie on how functions work. We wanted to compare the achievements of the students responding to my lectures when I used this visual and interactive medium, as opposed to their responses from the 'traditional' white-board lectures. To do this, we created a case study that compared 'traditional' and 'technology' focused teaching methods and asked the students similar questions about functions throughout the semester to gauge their understanding of the topic. Each test asked the students to give a definition of a function and then determine whether different scenarios represented functions as well as identify the domain and range for these scenarios. Since functions can be represented in many different ways, each test had a function listed as a statement, graph, table, and equation. This way we could track which representations caused the most difficulty.

DESCRIPTION OF THE METHOD: We started with a Pre-Test to one section of the Intermediate Algebra course at California State University, Channel Islands during the Fall semester of 2006. This was to be the baseline to compare the students' scores to after the lessons on functions were given. I then taught functions the 'traditional' way by lecturing on the white-board and using the examples out of the textbook, *"Elementary and Intermediate Algebra, Concepts and Applications Fourth Edition"* [1]. I asked them whether they had any questions before moving onto the next topic and assigned homework. Once we got through the chapter on functions, they took a quiz to see how well they understood the topic. Then after the midterm, I re-lectured on functions, but this time, I used the PowerPoint lesson that I created using the same examples out of the textbook [1]. I again asked whether they had any questions and assigned them homework. The next class period, they took another quiz to determine whether their scores improved.

After this, I interviewed two students that did not demonstrate an understanding of the function concept on any of the tests to evaluate where these students were struggling. The importance of understanding where these students struggled is that it may give educators some insight into tailoring future lectures on functions.

The experiments were conducted on students between 17 and 19 years old who were mostly freshmen at the university. Keep in mind that this is not the first time these students have been presented with the function concept. They have all had this material at least once before in High School and were expected to know Intermediate Algebra before entering the University. All of these students took an algebra test to evaluate their basic math skills and failed. These students were not interested in mathematics or the sciences. Most of them didn't show up to class and rarely attempted the homework. This made for a tough class to teach but it also exemplified the need for different approaches in math education. You will see that the improvements may have been small, but their interest in math increased significantly after the 'technology' lecture. That alone, was a breakthrough.

CHAPTER 2

Learning and Teaching Functions:

Regardless of what medium is used to teach functions, it is crucial to understand how students learn effectively and what steps are involved in the transition to higher level mathematics. We as human beings are only able to interpret and understand the world around us with our five senses: sight, hear, touch, taste, and smell. We are forced to rely on these senses when learning a new mathematical concept. Of these five senses, there are three effective ways of gathering and interpreting new information: visually, auditory, and kinesthetically. The research in mathematics education shows that we must use techniques that exercise all of these perception types in order to maximize the effectiveness of our instruction. Also we must not forget that a major part of etching mathematics is to get the students to do mathematics. In the essay *"The Nature of Mathematics: Its Role and Its Influence"* [2], John Dossey points out that "knowing mathematics is equated to doing mathematics ([2] page 44)."

According to David Tall in his essay *"The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof"* [7], there is an important point in a student's education when they are faced with a conflict between their expectations and reality. This conflict is one of the first parts of the transition to advanced mathematical thinking. A student must face and deal with their basic understanding of mathematics to allow the more general, and perhaps more difficult, concepts to be learned. The two main components of advanced mathematical thinking that students need to understand are precise definitions and logical deductions. The students must be ready to accept and understand the complete and precise definitions of a concept in order to understand it. Also, a student must be able to build logical deductions from these definitions to be able to handle all the variations of the concept. David Tall describes this transition as moving "from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal definitions and their properties reconstructed through logical deductions ([7] page 495)."

There are many cognitive changes involved in this transition that we must understand. They include: understanding a concept as a process; understanding that a concept is an object with a given name; the abstracting of properties from a definition; the constructing of the properties through logical deduction; and finally, the relationships between the concept and its various representations. All of these cognitive changes must be addressed when developing a lesson plan to teach a new concept. We need to focus on describing a concept and not just defining it. Definitions often make assumptions based on the student's concept image, their mental pictures and associated properties of their understanding of the concept. If there is a mismatch between the definition and the student's concept image, the definition can be obscured. To avoid this, we need to build cognitive roots. Cognitive roots are the building blocks of the student's concept image and allow for the new information by building off a solid conceptual foundation. These are often hard to find and require empirical research and mathematical knowledge to develop.

During this transition, there may exist a conflict between the student's old and new knowledge. This will create an obstacle to completely understanding the new concept. Note that a student's understanding of a concept depends on more than just the words used in the definition, their prior experiences will shape the way they read the definitions. As Ponce points out in his essay *"Critical Juncture Ahead!"*[6], the issue is "how to introduce the concept of functions to students in a way that taps into *their* prior knowledge and experiences ([6] page

138).” Therefore we need to know and understand the students’ prior experiences before we give them a new definition. More precisely, we need to give them this definition in ways that they are familiar with.

David Tall showcased the need to build more complete concept images in his essay [7] where he starts off by claiming that the set-theoretic definition of a function, Figure 1, can be difficult for students to understand. This definition is a good mathematical foundation, but is not a good cognitive root. That is why we often use alternate definitions like those in Figure 2 when teaching this topic. This trimmed down definition however seems to leave the students with an incomplete understanding of the true definition.

Figure 1

> Let A and B be sets, and let A x B denote the Cartesian product of A and B. A subset f of A x B is a function if, whenever (x_1,y_1) and (x_2,y_2) are elements of f_1 and $x_1 = x_2$ then $y_1 = y_2$,

Figure 2

> A function is a process which assigns to each element in one set (the Domain) a unique element in another set (the Range).
> Or,
> Each input has only one output

To test people’s understanding of what a function is, David Tall referenced a case study that asked high school and college students to determine whether a given formula or graph represented a function. He found many students thought that the formula $x^2 + y^2 = 1$ described a function simply because it was familiar to them. Also, most students regarded some graphs as not being functions not because they included vertical line segments, but because they looked strange. These examples show that high school and college students had problems with their concept image of functions. These results are largely duplicated in the case study described in this paper.

He goes on to suggest that there are ways to improve students’ understanding of this important topic. Technology, for instance, has been used to show the graphical representations of functions. However, this poses a risk to students only identifying functions as graphs. Unfortunately, the tools used to graph functions required entering them as a formula further emphasizing this ‘restriction’. The good thing with this approach however, is that it changes the idea of a function “from a rule-based, point wise process to a global visualization of overall behavior ([7] page 500).”

Serpil Konyalioglu also recognized that a more complete concept image is needed when teaching advanced mathematical concepts. She explains in *“The Role of Visualization Approach on Student’s Conceptual Learning”* [3] that using visualizations as part of the lecture dramatically helps students learn new concepts because they are able to interact with the concept in a variety of contexts. Some identities, statements, exercises, definitions and even theorems can be considered from various points of view to help a wider range of students’ progress in today’s curriculum. She points out that visualizations allow students to perceive relationships between abstract concepts and their underlying structures allowing them to make sense of the abstract concepts in mathematics. This understanding of the underlying structure allows students to successfully transition from a basic mathematical understanding to advanced mathematical thinking. They build a more complete concept image allowing them to handle the varying formats of the questions on tests.

Konyalioglu's results showed that visualizations provided the students with a way to understand the "abstract structures and concepts from a different perspective ([3] page 3)." The experimental students did much better on the conceptual questions and the final test results showed that these students were, on average, more successful than the control group students at the 0.05 significance level. She concluded that there was "not a meaningful difference between (the) procedural knowledge of (the) students…, (but) there is a meaningful difference between (their) conceptual knowledge ([3] page 5)." She explained that the success of the experimental students was a result of their ability to comprehend well intuitionally. Even though they received the same concept definitions, "they were exposed to different experiences which resulted in forming different concept image(s) ([3] page 5)." Ultimately, they had a concept image that was robust enough to answer questions that they had not seen before.

The 'traditional' mathematics teaching method is based on lecturing and mainly focuses on the development of the students' technical skills and usually does not emphasize the conceptual understanding of the underlying structure. Since at the algebra level most of the mathematical concepts become abstract, students trying to memorize them are no longer successful. Establishing the relationship between the student's knowledge and their intuition of these abstract structures may be an issue and introducing contemporary visualizations may help. Konyalioglu claims that "functional and permanent learning can be possible only by balancing conceptual and procedural knowledge ([3] page 2)." This is where well defined visualizations and related activities are needed to offset all the procedural knowledge that our students have been taught in lower level mathematics.

True progress in making the transition to higher mathematical thinking is made by making the student's reflect on their own thinking processes and confront the conflicts that arise. This is where visualizations and related activities are the most useful. They force this reflection and confrontation, and ultimately growth. Flexible work environments with open and free communication stimulate the process of building appropriate cognitive roots that help create a broader concept image. Current research shows that oversimplified environments designed to protect a student should be avoided since they tend to only complicate this transition by limiting the student's ability to deal with conflict. Additionally as Matej Mencinger points out in his essay *"On Some Visualizations at Different Levels of Mathematics Teaching"*[5], textbooks attempt to incorporate visualizations into them but "many authors of mathematical textbooks for secondary schools (colleges) are (partially) omitting the visualizations, because of the troubles related with printing and so on ([5] page 2)."

Following the above examples, this case study combines their strategies to show that by communicating to the students in a different way along with using visualizations and related activities in mathematics lectures, students will build a better concept image of functions and improve their problem solving skills. The research discussed in this chapter shows that visualizations produce a better concept image for the students and will ultimately help smooth the transition to higher level mathematics. This implies that all educators need to incorporate these contemporary techniques into their instructions to make sure that no students get left behind when abstract concepts are introduced.

CHAPTER 3

Case Study – Overview and Traditional Lecture:

This case study compares the performance of students exposed to two different presentation methods; the 'traditional' lecture based on reading the textbook [1] and lecturing using the whiteboard versus the 'technology' presentation based on computer visualizations. The study was conducted in one section of the Intermediate Algebra course at California State University Channel Islands (CSUCI) during the Fall semester of 2006. The course included mainly incoming freshman and all of the students failed a math placement exam prior to enrolling in this course. This class was designed as an alternate way of meeting a state required minimum mathematics competency for college level courses. Students had this material presented to them before in High School. In this section only 8 out of 30 students planned to major in a science related field, Biology and Environmental Sciences. Many students did not feel like they should have to take any more Mathematics courses since they were not majoring in the sciences. Interestingly, when asked how many hours a day they spent on their computers; the average response was 3 hours and several students claimed to be on their computers over 5 hours a day. Clearly, students today are very familiar with computers and rely on them to communicate with their peers.

This study was designed to support the hypothesis that students will respond more positively, learn more, and perform better on tests when taught functions using 'technology' versus a 'traditional' white board lecture. Students started with a Pre-Test to establish a baseline. Then I lectured in a 'traditional' way on functions following the examples in the textbook [1] and writing the definitions and working through the questions on the whiteboard. At the end of the class period, I assigned them homework out of the textbook [1] and then answered any of their questions at the beginning of the next class period. After completing all the lectures covering functions and their applications, I administered a quiz to determine whether their understanding of functions increased. Then after 4 weeks, the students took another test, the Midterm, to gauge their long term understanding of the function concept and related problem solving skills.

The next class period after the Midterm, I taught the students functions again using a visual PowerPoint presentation with animations and related activities [see Appendix G]. This was followed by a homework assignment and testing similar to the one after the 'traditional' lecture. Seven weeks later during the final exam, students worked on questions again about functions to check their long term understanding of the concept.

All the tests included comparable questions and were all graded consistently. Each test included defining a function and then asked to determine whether different scenarios represented a function. Appendix A through Appendix F contains all function related tests given during this study. The different representations of functions included; statements, graphs, formulas, tables, and pairs. Asking questions related to the different representations enabled me to understand which ones caused the biggest problems. All tests were graded using consistent rubric by assigning an integer value 0, 1, or 2 for each part of the question. A '0' meant that the student either left it blank, or they had no understanding of the concept. A '1' meant that the student had some, but not all, understanding of the concept. And finally, a '2' meant that the student demonstrated full understanding of the concept. Grading this way on all the tests, allowed me to track each student's progress on understanding the concept and to also identify which questions were the hardest to answer. Additionally, I graded the same question on all the students' tests before moving on to the next question so everyone's responses could be compared equally. The results of the case study showing comparisons of these tests are in Chapter 5.

Here is an example of the grading regime for the same question that was asked on every quiz: "Explain what a function is." If the student left this blank or didn't mention anything about a domain, range, or correspondence, they received a '0'. If they mentioned any of those components or any of the representations of a function (i.e. graph, table, machine …) but did not put all the pieces together correctly, they received a '1'. And finally, if they used all the right terms and put them together correctly, they earned a '2'. An example of a '2' from the Midterm was "A function contains an input and an output, and a domain and range. For every input there is only one output." (Skyler)

The rest of this chapter will focus on the first half of the case study, the 'traditional' lecture and its test score analysis, and the next chapter will discuss the 'technology' lecture and the analysis of its test scores. Questions asked at this level were to explain what a function is and then to determine whether two statements, an equation, a graph, and a table represented a function. The test is included in Appendix A. Analysis of the Pre-Test test scores is provided in Figure 3 below. The names used in the tables throughout this paper are not the true names of the students. Additionally, the first numerical row in these tables is the total possible points for each question.

Figure 3

Pre Test Analysis

	Describe	Statement	Statement	Statement Total	Formula	Graph	Table
Statistics	**Q2**	**Q3a Total**	**Q3b Total**	**Q3ab Total**	**Q3c Total**	**Q3d Total**	**Q3e Total**
Possible Points	2	8	8	16	4	4	4
Mean	0.25	1.39	1.64	3.04	0.75	0.86	1.61
Median	0	2	2	4	0	0	2
Mode	0	2	2	4	0	0	2
Standard Deviation	0.44	1.03	1.25	1.99	1.32	1.24	1.13
Minimum	0	0	0	0	0	0	0
Maximum	1	3	5	8	4	4	4
Count	28	28	28	28	28	28	28
Number of 0's	21	9	8	5	20	18	8
Number of 1's	7	1	0	0	1	0	0
Number of 2's	0	16	17	7	3	7	16
Number of 3's	-	2	1	1	2	2	3
Number of 4's	-	0	1	12	2	1	1
Number of 5's	-	0	1	0	-	-	-
Number of 6's	-	0	0	2	-	-	-
Number of 7's	-	0	0	0	-	-	-
Number of 8's	-	0	0	1	-	-	-
Number of 9's	-	-	-	0	-	-	-
Number of 10's	-	-	-	0	-	-	-
Number of 11's	-	-	-	0	-	-	-
Number of 12's	-	-	-	0	-	-	-
Number of 13's	-	-	-	0	-	-	-
Number of 14's	-	-	-	0	-	-	-
Number of 15's	-	-	-	0	-	-	-
Number of 16's	-	-	-	0	-	-	-

Question 2, denoted as Q2 in Figure 3, asked to define a function. The average score was only 0.25 points out of a possible 2, a 13% average and many students simply said that a function is "$f(x)$" or "an equation that solves for something." This confirmed that they have seen the topic before in the form of an equation. The seven students that got a 1 on this question wrote an answer like "one set transforms to another set" or drew a picture of a table with lines connecting the entries. These students showed that they knew some of the different pieces that make up a function, but could not put it together properly.

The next two questions asked to determine whether a statement represented a function and to explain why. They were also asked to identify the domain and range of the function. Question 3a asked whether the statement: "An insurance company assigns to every customer their phone number as an id number" represented a function. As expected, the students struggled here as well. The average score was 1.39 points out of a possible 8, a 17% average. Several students didn't think that this statement represented a function because there was no math to be done, or since there were no numbers in the problem. It appeared that many students were guessing because they left the explanation blank. Some of the students said that it was a function because every person's phone number is unique. This may have been a good argument, but not everyone has a phone number. Someone may buy insurance but doesn't own a phone (highly unlikely today, but it is still a possible scenario to consider). Then according to the statement, they would not get an ID number and thus this statement cannot represent a function. Also if someone had two different phone numbers than this would force one element of the domain to map to two elements of the range which is another reason this statement cannot represent a function.

Question 3b asked to determine whether the statement: "On official transcripts, CSUCI gives every graduating student his/her university GPA" represented a function. Out of a possible 8 points, the average score was only 1.64 points, a 21% average. Two students said that it was not a function because more than one student might have the same GPA. This implies that these students believed that the domain was the GPA (number from 0 to 4) and the range was the set of students. If the question were reversed, their argument would have been correct so they received 1 point. I would have given them a 2 on this if they stated the domain and range to support their reasoning. Like question 3a, there were many students that simply said 'Yes' but gave no explanation, implying that they guessed. There were also a few people that thought this statement was a function simply because math is involved in calculating a GPA.

The total combined average score for determining whether a statement represented a function was 3.04 points out of a possible 16, a 19% average, showing that students struggled with the concept. One of the reasons the average score was low was because most students left the domain and range portions of the question blank

Question 3c asked to determine whether the formula $f(x) = 3/(x\text{-}3)^2$ gives a well-defined function for all real numbers and to explain why. The average score for this question was 0.75 points out of a possible 4, a 19% average. Some people thought that it was a function for all real numbers simply because the formula started off with $f(x)$. Three students recognized that some real numbers could not be plugged into this equation because there is division involved, but they did not state that $x = 3$ was the problem. Others thought that was a function because you can plug in any number for x and get an answer. Overall, 5 out of the 28 students appeared to understand the relationship between a formula and a function.

Question 3d asked to determine whether a discontinuous graph represented a function and to explain why. The average score was 0.86 points out of a possible 4, a 21% average. Most people didn't think that the graph represented a function because there were two separate lines and they are not 'consistent'. Two students even went so far as to say that this could not be a function because it is not a straight line and it is curved. This may mean that these students had only seen linear graphing in their previous classes which would have clearly constricted their imaginations and this likely caused them to struggle with this abstract concept. Two students referred to the Vertical Line Test (VLT) which means they had some familiarity with the subject from their previous education.

Question 3e, asked to determine whether a given table represented a function and to explain why. This table related Horsepower to Top Speed. Two elements of the domain were assigned to one element of the range which may have confused the students. The average score on this was 1.61 points out of a possible 4, a 40% average, the highest so far. Some students thought that the table represented a function because the two columns were related 'realistically' or because it takes math to calculate horsepower. They noticed that as the horsepower increased, so did the Top Speed, thus this is a realistic scenario and must therefore be a function. Three students made the expected mistake where they thought that this was not a function because two numbers in the domain went to the same element of the range. Clearly, students were more familiar and comfortable with this representation of a function but still did not demonstrate full understanding of the definition.

After the Pre-Test, I taught them the function concept using the material straight out of the textbook [1] the 'traditional' way. This method is representative of how many mathematics instructors approach their lectures because they may rely heavily on the textbook. The 'traditional' method included these key activities:

- Asking the students to read the sections in the textbook [1]
- Encouraging student participation at all times
- Following the same outline of the material used in the textbook [1] to introduce new concepts
- Using the same definitions and examples in the textbook [1]
- Writing definitions and examples on the whiteboard
- Assigning homework out of the textbook [1] at the end of each class to be completed prior to the next class
    - All of the answers to the homework questions were in the back of the textbook [1] so students could check their work
- Starting off each lecture by answering any questions the students had on the homework or the last lecture before moving on to the next topic

The function concept was covered in sections 7.1 to 7.5 and 12.1 in the textbook [1] but the main skills used for identifying and describing functions were covered in sections 7.1 to 7.3. It took 4 class hours to cover the function concept the 'traditional' way. This also included the 15 minutes of questions at the beginning of each session. After 'traditional' lectures the students took a quiz to measure whether or not they developed the cognitive root necessary for this abstract concept. The quiz titled Chapter 7 Quiz but referred to as Quiz #1 in this paper is shown in Appendix B and it had questions similar to the Pre-Test. A summary of the analysis of the Quiz #1 test scores is provided in Figure 4 below.

Figure 4

Quiz 2 Analysis

	Describe	Statement	Statement	Statement Total	Formula	Graph	Table	Pairs	Domain (graph)	Range (graph)	Domain (equation)	Domain (pairs)	Range (pairs)	Domain (f+g)(x))	Domain (f/g)(x)
Statistics	**Q1**	**Q2a Total**	**Q2b Total**	**Q2ab Total**	**Q2c Total**	**Q2d Total**	**Q2e Total**	**Q2f Total**	**Q3c**	**Q3d**	**Q5b**	**Q6a**	**Q6b**	**Q9g**	**Q9h**
Possible Points	2	8	8	16	4	4	4	4	2	2	2	2	2	2	2
Mean	0.57	2.82	4.96	7.79	1.18	2.18	2.86	1.89	0.57	0.46	0.82	1.64	1.64	0.50	0.46
Median	1	3	6	9	1	2	4	3	0	0	1	2	2	0	0
Mode	1	3	6	9	0	2	4	0	0	0	0	2	2	0	0
Standard Deviation	0.57	1.56	1.57	2.83	1.31	1.42	1.63	1.73	0.79	0.74	0.77	0.62	0.62	0.69	0.58
Minimum	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
Maximum	2	6	6	12	4	4	4	4	2	2	2	2	2	2	2
Count	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28
Number of 0's	13	2	1	0	14	5	6	12	17	19	11	2	2	17	16
Number of 1's	14	4	0	1	0	3	0	0	6	5	11	6	6	8	11
Number of 2's	1	5	2	1	11	9	2	1	5	4	6	20	20	3	1
Number of 3's	-	8	1	1	1	4	4	9	-	-	-	-	-	-	-
Number of 4's	-	6	4	1	2	7	16	6	-	-	-	-	-	-	-
Number of 5's	-	1	4	2	-	-	-	-	-	-	-	-	-	-	-
Number of 6's	-	2	16	2	-	-	-	-	-	-	-	-	-	-	-
Number of 7's	-	0	0	1	-	-	-	-	-	-	-	-	-	-	-
Number of 8's	-	0	0	4						-		-	-		-
Number of 9's	-	-	-	8						-		-	-		-
Number of 10's	-	-	-	4						-		-	-		-
Number of 11's	-	-	-	1						-		-	-		-
Number of 12's	-	-	-	2	-	-	-	-	-	-	-	-	-	-	-
Number of 13's	-	-	-	0	-	-	-	-	-	-	-	-	-	-	-
Number of 14's	-	-	-	0	-	-	-	-	-	-	-	-	-	-	-
Number of 15's	-	-	-	0	-	-	-	-	-	-	-	-	-	-	-
Number of 16's	-	-	-	0	-	-	-	-	-	-	-	-	-	-	-

Students improved on describing a function as compared to their Pre-Test scores and the average score this time was 0.57 points out of a possible 2, a 29% average. Only one person, Lauren, gave a correct definition and received 2 points. Fourteen other students received a score of 1 because they referenced different attributes of a function like domain, range, passing the VLT, and function machine. Some students also referenced one output for every input and others mentioned the words "set" and "process." The remainder of the students did not get close to explaining what a function is and received a score of 0. We quote here some of their answers: "Something that only represents one thing" Amanda; "relationship between two sets of numbers" Brittany; "ordered pairs, or consecutive numbers" Denise; "set of numbers that have a correspondence" Matt; and "equation that has the correspondence of two variables" Ryan. Even though the average score increased from the Pre-Test, many students still struggled to describe a function.

The next question asked to determine whether the statement: "The correspondence that assigns to each person of a Rock Band the instrument that they can play" represented a function. Note that since someone can play more than one instrument this is not a function (because an element in the domain would correspond to more than one element of the range). The average score for Question 2a was 2.82 points out of a possible 8, a 35% average. About one third of the students believed that this would be a function because they assumed a person can only play one instrument and therefore these students received only partial credit. More than two thirds of the students correctly identified the elements of the domain and range, but did not properly identify the whole domain and range. They simply answered "person" for the domain, whereas the correct answer is "the set of people in the Rock Band." Two students correctly stated that it was not a function but for the wrong reason. They thought that more than one person may play the same instrument (this could be a justified answer if they switched the domain and range but they stated "person" was the domain, and "instrument" was the range). Since this was not complete reasoning, they only got partial credit. One student said that it was not a function because there weren't any numbers and two students didn't think that there was a domain or range because this was not a function. Interestingly, two students said the domain and range was all real numbers, even though no numbers were mentioned in the question, and four others said the domain was "Rock Band" instead of members of the band. One student, Amanda, made the realization that if someone could play more than one instrument then this would not be a function, so she limited the domain to people that could only play one instrument, hence she received full credit for her answer. Another student, Maura, switched the domain and range and claimed that this would be a function because a "person (can) play only one instrument." However, this statement would still not be a function because more than one person might play the same instrument in the band, so she only received partial credit.

Question 2b asked to determine whether the statement: "The correspondence between a person and their birthday" represented a function. The average score was 4.96 points out of a possible 8, a 62% average. This is significantly higher than the previous questions because most students realized that each person can only have one birthday. Four out of the 28 students did not believe that this statement is a function. Two of those students read the question backwards, but were still able to answer it correctly based on their domain and range. They thought that the domain was "Birthday" and the range was "People" and then correctly recognized that some people have the same birthday, thus breaking the uniqueness rule. The same two students that said the domain and range was all real numbers for Question 2a also said the domain and range for this statement was all real numbers clearly showing that they struggle to understand how the definition of a function applies to non-numerical relationships. Also, more than 85% of the students correctly identified the elements of the domain and range, but did not properly identify the whole domain and range; i.e. they simply answered "birthday" for the range, whereas the correct answer should be "the set of days in the calendar."

The total combined average score for determining whether a statement represented a function on Quiz #1 was 7.79 points out of a possible 16, a 49% average, showing that these students still struggled. Several students appeared to grasp the concept better but as a group, they still did not demonstrate full understanding.

Question 2c asked to determine whether the equation $x^2 + y^2 = 1$ represented a function and to explain why or why not. Since this is the equation of a circle this is not a function. But students may have not necessarily seen this equation before. The average score for this question was lower than expected at only 1.18 points out of a possible 4, a 29% average which was a slight improvement as compared to the similar question on the Pre-Test. Many students thought that this formula represented a function but their reasons why varied greatly. Some students gave simple reasons based on the fact that the formula had an x and a y in it. Another student, Maura, said that it was a function because it is of the form "Ax + By = C." This answer indicates that only formulas she was comfortable with were linear. One student even said that it was not a function because there was not enough information, Andrew. One of the students said that it was not a function because of the "vertical line test", Skyler, which implies that he did recognize this formula as a circle.

Question 2d asked to determine whether a continuous graph with a point removed from it and added below the hole represented a function. A simple application of the VLT could show that this indeed was a function. Many students used the VLT and concluded the graph represents a function. The average score was 2.18 points out of a possible 4, a 54% average, hence a noticeable improvement as compared to the Pre-Test. But still four students struggled on this question. One student, Andrew, said that this was a function because "it has a domain and range on the graph." Another student, Atylana, said that it was not a function because "the line crosses the x & y axis more than once."

In the next two questions, students studied a table and a series of pairs. Question 2e, a table, and Question 2f, a series of pairs, showed one input with two different outputs, implying that neither represented a function. The average score for Q2e was 2.86 points out of a possible 4, a 71% average, showing that the students were comfortable with tables. Q2f generated the average score of only 1.89 points out of a possible 4, a 47% average. Pairs were not on the Pre-Test but were explained in the lectures. It appears that the students struggled with this notation and probably would have scored higher if they placed the pairs into a table. Three students argued that the table represented a function because every element of the domain went to at least one element of the range which defines a relation but not necessarily a function and therefore they received partial credit. Curiously, one of these students, Adrianne, noticed that the pairs did not represent a function because when graphed they fail the VLT but she did not connect this fact that one input produced two outputs back to the table hence answered that question incorrectly. Four other students didn't see the connection between the two representations either and made similar mistakes. Two students correctly identified Q2e as not a function, but for the wrong reason. They thought it was not a function because two elements of the domain went to the same element of the range, a pervasive error. One of the students, Matt, thought that Q2f was a function because there were "no duplicate ranges" and another student, Andrew, said that both Q2e and Q2f were functions simply because "it gives you domain and the correspondence to the range" which describes a relation but not a function. Two students, Malloree and Tiffany, said that Q2f was a function because each value listed "is a point" and another student, Robert, thought that Q2f was a function because "the set of points are all different." At this point students formed different opinions about functions, but their cognitive roots were not sufficient to handle the different representations.

The final series of questions on the first Quiz (Q3c, Q3d, Q5b, Q6a, Q6b, Q9g and Q9h) asked to identify the domain or range for different representations of functions. These questions helped determine how well the students understood two of the key building blocks of a function. Questions 3c and 3d asked for the domain and range, respectively, of a continuous graph defined on a closed interval (i.e. with two closed end-points). The average score for Q3c was 0.57 points out of a possible 2, a 29% average, and the average score for Q3d was 0.46 points out of a possible 2, a 23% average. Only five students answered correctly using the proper notation in their answers. At this point in the course students were taught only set notation and this is what most students used in their answers. However, two students, Adrianne and Atylana, used interval notation, even though they didn't learn that in this class. This situation confirms that several students had studied functions before. Some students listed the end-points to represent the domain and range, whereas others simply put all real numbers. This time only one student had the domain and range backwards and three other students had the correct sets listed, but did not use the proper notation.

Question 5b asked for identifying the domain of a function given by a formula with $(x+5)$ in the denominator implying that $x \neq {}^{-}5$. The average score for this question was higher than expected at 0.82 points out of a possible 2, a 41% average. Six people answered the question correctly and stated "$x = {}^{-}5$ is a problem." Eleven people got close, but could not identify the correct value of x that was not in the domain and received partial credit. Some of those students thought $t = {}^{-}3$ was the problem, suggesting that they were finding the domain of $g(t\text{-}2)$ from question 5a and not of g itself. Three students incorrectly said the domain was all real numbers.

The next questions asked for the domain and range of a set of pairs. The students appear to be most familiar with this notation as they averaged 1.64 point out of a possible 2, an 82% average for both Questions 6a and 6b. This is the first time when the majority of the students provided the correct answers. Only two students didn't get even partial credit and one other, Alex, said that the domain and range was all real numbers even though the sets contained symbols ‡ and @. Six students listed the correct elements of the domain and range, but did not use the proper notation and received partial credit. They used either square brackets or parenthesis instead of the curly brackets to indicate a set of values.

Questions 9g and 9h asked for the domain of a function defined by a formula resulting from combining two other formulas. More specifically Q9g asked for the domain of $(f+g)(x)$ and Q9h asked for the domain of $(f/g)(x)$ (note that the domain can be interpreted as the common part of the domains of f and $1/g$). The students struggled and the average score for Q9g was 0.50 points out of a possible 2, a 25% average, and the average score for Q9h was 0.46 points out of a possible 2, a 23% average. Only one student answered both questions correctly and two others answered Q9g correctly, but not Q9h. Twelve students were close to answering the questions, but could not correctly identify the values to remove from the set of real numbers, and thirteen other students simply answered "all real numbers."

After four weeks, students were asked questions about functions on their Midterm to evaluate their long term understanding of the concept. This test, like all the other tests, was closed book and the function questions were similar to the ones in the Pre-Test and Quiz #. Two new extra credit questions tested the development of their cognitive roots related to the definition of a function. (Note there were two students that took the Midterm that did not take the Pre-Test or the first Quiz. We removed them from the comparative analysis in Chapter 4). The complete list of function related questions given on the Midterm are in Appendix C and the analysis of these test scores is provided in Figure 5 below.

Figure 5

Midterm Analysis

	Describe	Statement	Statement	Statement Total	Pairs	Range (pairs)	Formula	Graph	Range (graph)
Statistics	Q1	Q2a Total	Q2b Total	Q2ab Total	Q2c Sub-Total	Q2c3	Q2d Total	Q2e Sub-Total	Q2e3
Possible Points	2	8	8	16	4	2	4	4	2
Mean	0.77	2.93	2.17	5.10	2.73	1.37	1.53	1.87	0.23
Median	1	3	2	5	3	2	2	1.5	0
Mode	1	2	2	4	2	2	0	4	0
Standard Deviation	0.63	1.44	1.32	2.04	1.11	0.76	1.41	1.55	0.50
Minimum	0	0	0	1	0	0	0	0	0
Maximum	2	6	5	9	4	2	4	4	2
Count	30	30	30	30	30	30	30	30	30
Number of 0's	10	1	4	0	2	5	12	7	24
Number of 1's	17	2	2	2	0	9	1	8	5
Number of 2's	3	11	16	0	11	16	8	5	1
Number of 3's	-	6	3	4	8	-	7	2	-
Number of 4's	-	6	3	8	9	-	2	8	-
Number of 5's	-	2	2	3	-	-	-	-	-
Number of 6's	-	2	0	3	-	-	-	-	-
Number of 7's	-	0	0	7	-	-	-	-	-
Number of 8's	-	0	0	2	-	-	-	-	-
Number of 9's	-	-	-	1	-	-	-	-	-
Number of 10's	-	-	-	0	-	-	-	-	-
Number of 11's	-	-	-	0	-	-	-	-	-
Number of 12's	-	-	-	0	-	-	-	-	-
Number of 13's	-	-	-	0	-	-	-	-	-
Number of 14's	-	-	-	0	-	-	-	-	-
Number of 15's	-	-	-	0	-	-	-	-	-
Number of 16's	-	-	-	0	-	-	-	-	-

	Table	Domain (table)	Domain (equation)	Domain (equation)	Domain (pairs)	Range (pairs)	Formula	Graph (How Make)	Statement (How Make)
Statistics	Q2f Sub-Total	Q2f3	Q3	Q4	Q7a	Q7b	Q8f Total	EC1	EC2
Possible Points	4	2	2	2	2	2	4	2	2
Mean	1.80	0.93	0.70	0.83	1.63	1.37	1.20	0.50	0.63
Median	1	1	0.5	1	2	1.5	0.5	0.5	0.5
Mode	0	1	0	1	2	2	0	1	0
Standard Deviation	1.90	0.69	0.79	0.70	0.72	0.72	1.30	0.51	0.72
Minimum	0	0	0	0	0	0	0	0	0
Maximum	4	2	2	2	2	2	3	1	2
Count	30	30	30	30	30	30	30	30	30
Number of 0's	15	8	15	10	4	4	15	15	15
Number of 1's	0	16	9	15	3	11	1	15	11
Number of 2's	2	6	6	5	23	15	7	0	4
Number of 3's	2	-	-	-	-	-	7	-	-
Number of 4's	11	-	-	-	-	-	0	-	-
Number of 5's	-	-	-	-	-	-	-	-	-
Number of 6's	-	-	-	-	-	-	-	-	-
Number of 7's	-	-	-	-	-	-	-	-	-
Number of 8's	-	-	-	-	-	-	-	-	-
Number of 9's	-	-	-	-	-	-	-	-	-
Number of 10's	-	-	-	-	-	-	-	-	-
Number of 11's	-	-	-	-	-	-	-	-	-
Number of 12's	-	-	-	-	-	-	-	-	-
Number of 13's	-	-	-	-	-	-	-	-	-
Number of 14's	-	-	-	-	-	-	-	-	-
Number of 15's	-	-	-	-	-	-	-	-	-
Number of 16's	-	-	-	-	-	-	-	-	-

The first question on the midterm was the exact the same question that was on the Pre-Test and Quiz #1. The students had seen it several times already. This question asked for a definition of a function and the average score was still only 0.77 points out of a possible 2, a 38% average. This is an increase from a 0.25 average from the Pre-Test and a 0.57 average from Quiz #1, but still the majority of the students could not properly define a function. The number of 0's dropped from 21 in the Pre-Test and 13 in the first Quiz to 10 on the Midterm. Three students actually scored 1 on the Pre-Test and 0 on the Midterm, whereas seven students increased their score from a 0 to a 1. There was also an increase in the number of 2's from 0 in the Pre-Test and 1 in Quiz #1 to 3 students giving the correct definition on the Midterm. Some of the answers that were marked as a 0 showed some understanding of the concept, but their cognitive root was not completely developed and they struggled to put all the necessary pieces together. Here are examples of their answers: "set of numbers with something representing something else" Amanda; "set in which numbers will correspond to each other" Andrew; "when a point crosses only once" Atylana; "set of numbers and symbols that work together to come to a solution" Max; "equation that can be graphed or solved for a given set of numbers" Robert; and "equation that has two variables that both add to a sum" Ryan.

The next two questions asked to determine whether a statement represented a function, find domain and range, and explain their thinking. Question 2a asked whether the statement: "The correspondence that assigns a price to a bar code on a product in a store" represented a function. In order to get the correct answer for this question, the student had to set up the domain and range correctly and state that each barcode only corresponds to 1 price. Even though some of them switched the domain and range, they correctly explained that some product's barcodes have the same price and therefore this was not a function, hence they received full credit. The average score for Q2a was 2.93 points out of a possible 8, a 37% average. Twenty-seven students answered 'yes' and of these, thirteen switched their domain and range around and should therefore have answered 'no.' Two students switched the domain and range but did correctly state that this was not a function because some products cost the same. In summary, 16 out of 30 students answered and supported their answer correctly. All but three students got close to identifying the domain and range correctly by stating they were simply 'barcode' and 'price' respectively.

Question 2b reads "A bank uses your dog's name as the answer to a security question in case you forget your password. And if you have more than one dog, they combine their names to form one larger name. Does this process describe a function?" This question is challenging as some customers may not own a dog. If the student identified the domain as the set of customers and the range as the set of names of all the customer's dogs, then this does not represent a function since not everyone owns a dog. The average score for Q2b was 2.17 points out of a possible 8, a 27% average. Twenty students correctly identified this as not a function; however, twelve of those students could not support their answer. Some students struggled with the scenario of a customer having more than one dog, indicating they did not understand the process. For instance, one student claimed this statement did not represent a function because "you can't combine the dog's names", Alexandria. Another student, Robert, read the process as combining names and not answering a security question. He used the "number of dogs" as the domain and "the length of name" as the range. Then he correctly stated that this was a function because "the name changes in turn with the number of dogs." Since his argument was solid, he got a 1 on this question. Two students switched the domain and range and eight others got close to identifying the domain and range. Sadly, four students were not able to get any part of this question correct.

The total combined average score for determining whether a statement represented a function on the Midterm was 5.10 points out of a possible 16, a 32% average, showing that these students still struggled to explain why a statement represented a function. This number actually decreased from Quiz #1 (a 7.79 average). One student, Alan, claimed that the domain and range was all real numbers for both of these statement problems even though no numbers were mentioned in the question. This student also gave this same answer as the domain and range for the statements from Quiz #1 and appears to be struggling with the conflict that a function does not have to contain numbers.

Question 2c asked whether a series of pairs represented a function and also to identify the range. The students scored better on determining whether the pairs represented a function, 2.73 points out of a possible 4, while on the first Quiz they only averaged 1.89 points out of 4. The average score dropped to 1.37 points out of a possible 2, a 68% average, on the range part of the question. One student, Alan, said the range was all real numbers even though the symbol "#" was in the range. Another student, Alexandria, said that this was not a function because "they don't correspond"; however she correctly identified the range. One student, Darren, correctly justified why the pairs represented a function but listed all the elements of the domain and range when only the range was asked for and therefore only received 1 point for this part of the question.

Question 2d asked to determine whether the formula $y = \pm 5x$ represented a function. The challenge for the students was to understand the ± symbol and notice that this will give two outputs for every input and therefore not a function. The average score was only 1.53 points out of a possible 4, a 38% average, and comparing this to the similar low scores on the Pre-Test and Quiz #1, the students still struggled with this type of question. Five students believed that this was a function because it can be graphed and one of these students, Gabby, drew the graph and claimed that it passes the VLT; even though when she drew a vertical line, it crossed the graph twice and failed the VLT. Nine students recognized that the ± symbol caused a problem and therefore stated that this was not a function. One student, Brittany, was very honest with her thoughts and said that it was a function "because I am guessing and one of the two is right."

Question 2e showed a graph of a continuous function and one extra point (a closed dot) above the graph to signify that there are two outputs for one input. The modified graph did not represent a function. The average score for this question was 1.87 points out of a possible 4, a 47% average; and when compared to the Pre-Test score of 2.18 points, the students actually performed slightly worse. Six students claimed that this graph failed the VLT. Eight students claimed that this passed the VLT and mysteriously one of them, Cole, even drew a vertical line that connected the two 'dots' indicating that this should have failed the VLT. This indicated these students understood that the VLT was a good way to test whether a graph represented a function, but they did not know how to use the VLT properly. Two students claimed that this was a function because it only crosses the x-axis once. This is a valid test for linear graphs and students' experience with non-linear graphing was still limited and their concept image was not complete.

The second part of Question 2e was to determine the range of this graph. The average for this question was a meager 0.23 points out of a possible 2 points, a 12% average. Five students believed that the range was all real numbers, the default answer, and four other students said that the range was (1, 2.2), which was the added point above the main part of the curve. Four students successfully "framed" in the range as was taught in class, but did not consider the arrows at the end of each line that are used to show that this pattern continues passed the shown graph and therefore these students received 1 point.

Another table representation was included in the next question and it was not a function. The average score for Question 2f was 1.80 points out of a possible 4, a 45% average. The students scored about the same on the Pre-Test as they did on the Midterm and much lower than they did on Quiz #1 where they averaged 2.86 points. The students' responses indicate that they did not recognize that a function requires ever element of the domain to go to exactly one element of the range. Thirteen students claimed that this was a function because no input went to more than one output. But the condition that they did not recognize was that to represent a function, every input has to go to exactly one output.

A sub-question to Question 2f was to identify the domain of that table and the average for this question was similarly low at 0.93 points out of a possible 2 points, a 47% average. Fourteen students simply described the domain and did not list the elements. Three students switched the domain and range and one student, Alan, again claimed the domain was all real numbers.

Questions 3 and 4 asked to determine the domain of a function defined by an equation. A rational function was given in Question 3 and the average score was 0.70 points out of a possible 2, a 35% average. Four students set the numerator equal to zero instead of the denominator and two other students simply said that the domain was all real numbers. Four students claimed that the domain was "3*t*-9", the denominator, and five students correctly identified that x cannot equal 3 and needs to be removed from the domain but did not use the correct notation in their answer. Question 4 included a square root and the average score was 0.83 points out of a possible 2, a 42% average. Ten students claimed the domain was all real numbers except 6 when in fact that the domain was all real number greater than or equal to 6 and four students simply said the domain was all real numbers.

Question 7 asked to determine the domain and range for a list of pairs where two values of the domain corresponded to a single number in the range. The average score for identifying the domain was 1.63 points out of a possible 2, an 82% average, and the average score for the range was 1.37 points out a possible 2, a 68% average. The majority of students listed the elements of the domain correctly, and 3 other students listed the correct elements but did not use the proper notation. Fifteen students correctly identified the range of these points, and nine students made the mistake of listing "7" twice in the range. Two students misunderstood the question and separated the points into two sets, domain and range, and then listed the set intersection of the points as the domain and the union of the points as the range.

Questions 8f asked to determine whether the formula $(f/g)(x)$ represented a function and to explain why or why not. An added challenge came from the fact that students had to combine two functions correctly first. The average score for this question was 1.20 points out of a possible 4, a 30% average. Seven students correctly stated that this was not a function because $g(x) = x + 3$ cannot equal 0 but did not fully explain why that made $(f/g)(x)$ not represent a function. Seven different students claimed that this was a function because you can plug in any real number for x and get a real number answer, even though this is not true when $x = -3$. Two students recognized that division was a problem but did not elaborate on that. One of these students creatively answered that this was not a function for all real numbers because "fractions are bad", Amanda.

In addition, new extra credit questions were introduced to evaluate the development of their cognitive roots and to stretch their understanding. The first extra credit question showed a graph that included what appeared to be 'vertical' segments. For this graph to be a function, the segments that look vertical would have to actually not be vertical. Thirteen students got close to the correct answer and said that this graph could not be a function because it failed the VLT.

Based on their responses, the average score for this question was 0.50 points out of a possible 2, a 25% average.

The second extra credit question re-visited Question 2a from Quiz #1 in a modified version and asked how to change the correspondence that assigns to each person of a Rock Band the instrument that they can play into a function. "No one in the Rock Band can play more than one instrument" was chosen by eleven students in their responses. The average score for this question was 0.63 points out of a possible 2, a 32% average. One student, Matt, stated that this statement could represent a function if "we could set a limit to the amount of instruments available"; however setting a limit on the range does not exclude someone from being able to play more than one instrument.

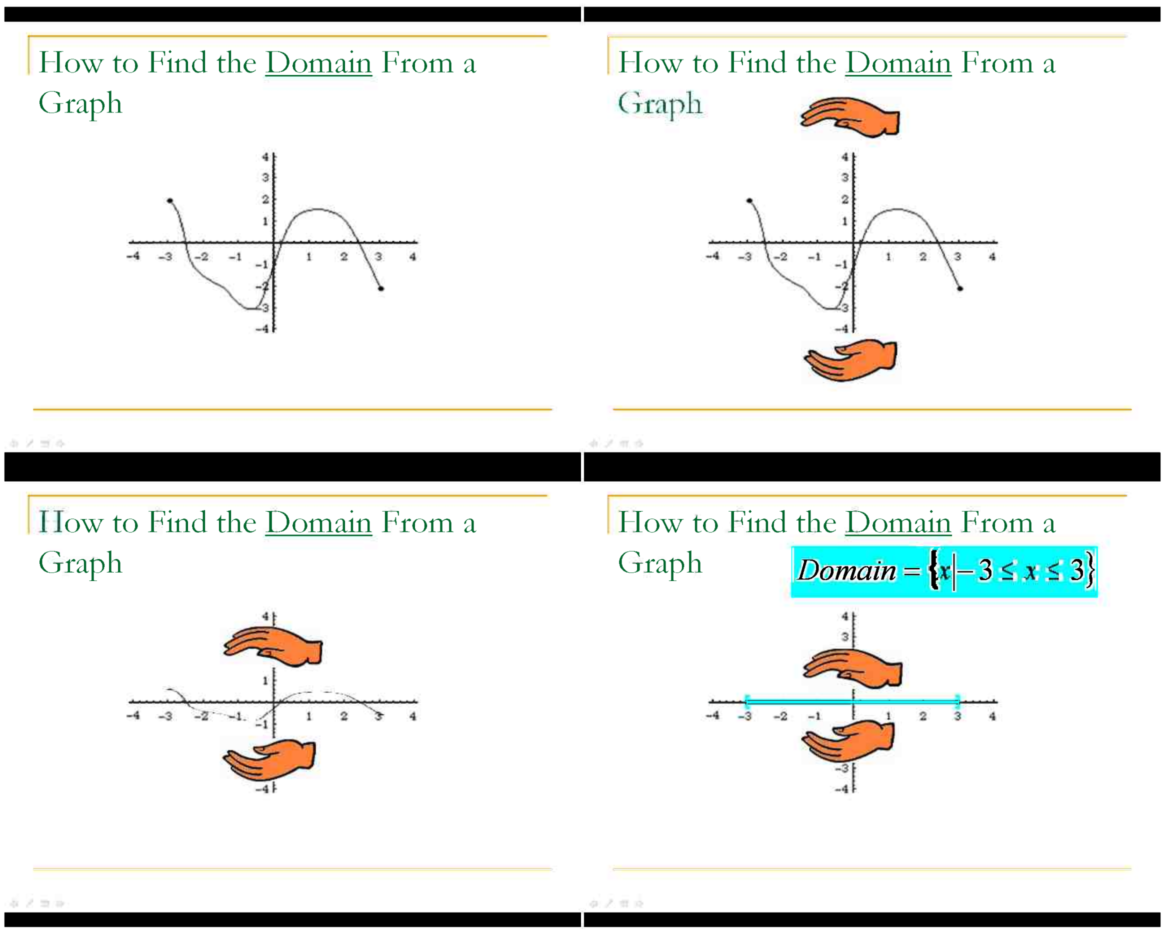
CHAPTER 4

Case Study – Technology Lecture:

The day after the midterm and before getting their tests back, students participated in a specifically designed technology based lecture on functions. The PowerPoint presentation followed the textbook [1] and presented topics in the same order. Presenting the material using this visual and interactive mode was designed to develop new cognitive roots necessary to understand the concepts.

My PowerPoint presentation contained many animations and had a hyperlink to an applet [8] for function machines. All definitions and theorems were written clearly, legibly and logically. Some of the graphics were taken from the Internet and some others I designed using Microsoft Paint. This presentation was something that any teacher could create with little experience. One of the animations showed how to find the domain of a graph by 'squashing' it down onto the *x*-axis (Figure 6 below shows four screen-shots taken during the animations). This technique came from the textbook [1] and I tried to show this on the whiteboard during the 'traditional' lectures.

Figure 6



We started by defining a relation, including the domain and range, and then a function. After this, we worked with function machines and used several animations to help convey the concept. The Function Machine applet [8] was a fun visualization and we did not use numerical values to stress that functions can be defined for various sets. Next we studied the graphical representation of functions and used animations to determine the domain and range from a graph by 'squashing' it down onto the *x*-axis for domain and onto the *y*-axis for the range. We used both graphs with and without end points, and with and without 'holes', to give the students a robust understanding of the technique. We also visualized the VLT on various graphs, deciding whether they were functions. After graphing, we re-visited the definitions and checked whether or not various statements represented functions and determined the domain and range. Lastly, we worked with formulas and whether or not they represented functions for all real numbers. We worked through a couple of examples that included quotients and roots and we summarized with a review of the major topics and a question session.

Note the 'traditional' way took 4 hours of instruction time. In contrast, this visual presentation only took 1.5 hours. Since the material was created before class, there was no need to write on the whiteboard. This allowed more time to interact with the students and answer their questions. I also used a hand held presentation device with a laser pointer so that I was free to roam around the classroom and face the students. Also, the laser pointer allowed me to highlight points of interest without having to stand by the projection screen. It is worth to report that students were very involved and really enjoyed the visual presentation and were interested every time an animation started. Sometimes, they even asked to see them again. If I tried to move on to the next slide too fast, they would stop me and ask to go back. This rarely happened during 'traditional' lectures.

Parts of the presentation were given as a handout so the students could take notes during class. Most of these students didn't take notes in previous class sessions and this way they had a handout to review later. Homework was attached at the end of the notes and contained questions similar to the ones they had answered before. The complete set of slides from the PowerPoint lecture and the corresponding copy that was handed out to students for notes is contained in Appendix G.

Two versions of Quiz #2 were given to only those students who participated in the 'technology' lecture during the next class session. Students who performed poorly on the Midterm worked on the Yellow Quiz #2. This quiz contained questions similar to the Pre-Test, Quiz #1 and the Midterm [see Appendix D]. Students who performed well on the Midterm and have shown understanding of the function concept worked on the Blue Quiz #2. This quiz contained more difficult conceptual questions similar to the Extra Credit questions from the Midterm [see Appendix E]. The analysis of the Yellow quiz test scores is shown below in Figure 7. The analysis of the Blue quiz test scores is given in Chapter 5.

Additionally on both quizzes, students answered a couple of qualitative questions to determine how familiar they were with computers and whether or not they preferred the 'technology' lecture. These questions were:

- Did you take notes during the last lecture?
- Did you take notes in the previous lectures?
- Did you feel more comfortable during the last lecture or the previous lectures?
- Which types of notes do you like better, your own or the prepared ones?
- Do you feel that you know more now after having the PowerPoint lesson?
- How much time do you spend on the computer on an average day?

Figure 7

Quiz #2 (Yellow) Analysis

	Describe	Statement	Statement	Statement Total	Formula	Graph	Range (Graph)	Table	Domain (Table)	Range (Table)
Statistics	**Q1**	**Q2a Total**	**Q2b Total**	**Q2ab Total**	**Q2c Total**	**Q2d Sub-Total**	**Q2d3**	**Q2e Sub-Total**	**Q2e3**	**Q2e4**
Possible Points	2	8	8	16	4	4	2	4	2	2
Mean	0.87	3.87	4.20	8.07	2.07	1.53	0.60	2.60	1.27	1.27
Median	1	3	5	8	2	1	0	4	1	1
Mode	1	3	5	5	3	1	0	4	1	1
Standard Deviation	0.52	2.23	1.57	2.79	1.49	1.36	0.74	1.80	0.46	0.46
Minimum	0	0	2	5	0	0	0	0	1	1
Maximum	2	8	6	14	4	4	2	4	2	2
Count	15	15	15	15	15	15	15	15	15	15
Number of 0's	3	1	0	0	3	2	8	3	0	0
Number of 1's	11	0	0	0	3	9	5	3	11	11
Number of 2's	1	2	4	0	2	1	2	0	4	4
Number of 3's	-	6	1	0	4	0	-	0	-	-
Number of 4's	-	2	1	0	3	3	-	9	-	-
Number of 5's	-	0	6	4	-	-	-	-	-	-
Number of 6's	-	2	3	1	-	-	-	-	-	-
Number of 7's	-	0	0	2	-	-	-	-	-	-
Number of 8's	-	2	0	2	-	-	-	-	-	-
Number of 9's	-	-	-	2	-	-	-	-	-	-
Number of 10's	-	-	-	1	-	-	-	-	-	-
Number of 11's	-	-	-	1	-	-	-	-	-	-
Number of 12's	-	-	-	1	-	-	-	-	-	-
Number of 13's	-	-	-	0	-	-	-	-	-	-
Number of 14's	-	-	-	1	-	-	-	-	-	-
Number of 15's	-	-	-	0	-	-	-	-	-	-
Number of 16's	-	-	-	0	-	-	-	-	-	-

The first question on the Yellow Quiz #2 asked to define a function (as on the Pre-Test, Quiz #1 and the Midterm). The data showed that the average score for Question 1 increased from 0.57 out of a possible 2 from Quiz #1 to 0.87 points on Quiz #2, a 43% average. The number of 0's decreased from thirteen on Quiz #1 to only three on this quiz. One of the students that scored a 0 simply said that a function is a "relationship between one property and another", Ryan. The other student that got a 0 on this question stated that a function is "the number of times a line or graph crosses the *x*-axis", Atlyana. Even after the 'technology' lecture, some of them still did not have a complete understanding of functions.

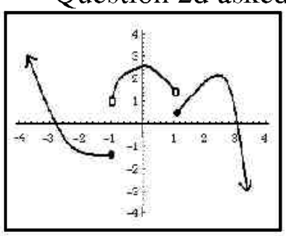
The next two questions asked to determine whether a statement represented a function and to identify the elements of the domain and range. Question 2a to determine whether the statement: "When a movie director casts roles for parts in their movie, they assign an actor to each character in the screenplay" represented a function. Some students used the set of actors as the domain and the roles in the movie as the range, hence not a function because an actor may have more than one role in a movie. Some switched the domain and range and stated that this was a function because for each role in the movie, there is only one actor who can play the part, and received full credit. The average score was 3.87 points out of a possible 8, a 48% average. Eleven students stated that each role only has one actor that can play the part, but only two of them identified the domain and range properly to support their answer. Two students stated that this was not a function because an actor may play more than one role in the film. One student, Alan, again stated that the domain and range was all real numbers even though there were no numbers mentioned in this question.

Question 2b asked "If you make $10 an hour and you are not allowed to work overtime, does the correspondence between the number of hours you work and the amount of money you make describe a function?" The average score was 4.20 points out of a possible 8, a 53% average. One student, Joel, stated that this was not a function because "you can make $30 an hour and still work overtime." Alan again stated that the domain and range included all real numbers, but in this case, he was almost right because hours worked and total pay can both be expressed as real numbers. Therefore he received a 1 for each of these questions.

The total combined average score for determining whether a statement represented a function on the Yellow Quiz #2 was 8.07 points out of a possible 16, a 50% average. This was the highest average so far, but it was clear that students still did not fully understand the function defined by statements. One student, Alan, has claimed all real numbers as the domain and range for every statement question asked since the first Quiz hence still not understanding these concepts.

Question 2c asked to determine whether a given formula represented a function for all real numbers and to explain why or why not. The formula had $(x - 5)^2$ in the denominator. The average score was 2.07 points out of a possible 4, a 52% average. Seven students identified an issue when $x = 5$ and correctly stated that this was not a function for all real numbers. Two students knew how to tell whether a formula was a function for all real numbers as follows: "any number put in for x makes a real number answer" Cole. They did not recognize that this is not true when $x = 5$ therefore they only received 1 point for this. It appears that the students improved understanding of definitions by formulas as their average scores increased from 1.18 points on Quiz #1 to 1.53 points on the Midterm to 2.07 points on Quiz #2.

Question 2d asked to determine whether the following graph represented a function. This



type of question has been a problem for the students in the past. The average score was 1.53 points out of a possible 4, a 38% average which shows that the students still struggled with non-linear graphs that have discontinuities in them. The average score from Quiz #1 was 2.18 points and was only 1.87 points on the Midterm for similar questions. Ten students used the VLT in their answer, but seven of these students claimed that it fails the VLT. Two students claimed that there is more than one output when $x = -1$ and $x = 1$ indicating that they did not understand the notation. Two other students simply stated that this was not a function because the lines are not connected. Most students understood the VLT was a good way to tell whether graphs represent functions but they didn't know how to use it when there are breaks in the graph. Most students struggled with the notation of using open and closed dots.

Additionally, on identification of the range of the graph in Q2d the average score was 0.60 points out of a possible 2, a 30% average. Three students claimed that the range was all "y" values between 3 and -3. This would be true if there were not arrows on the end of the graph indicating that they continue to infinity. One student gave the range for each of the 'pieces' of the graph, and another student listed the range as the set of y-values that correspond to the open and closed 'dots'. When asked similar questions on the first Quiz and on the Midterm, the students also struggled with the range of a graph. They averaged 0.46 points on Quiz #1 and 0.23 points on the Midterm.

A table that relates the Time of Day with the Temperature was used in the next question, Question 2e. Since there was no output Temperature at 2:00pm, this table did not represent a function. The average score for this question was 2.60 points out of a possible 4, a 65% average. This was an improvement over the Midterm where they averaged 1.80 points. Nine students recognized that 2:00pm did not correspond to any Temperature and correctly stated that this therefore did not represent a function. Three students erroneously stated that each element of domain only goes to one element of range making this table a function. Question 2e also asked to identify the domain and range and the average score was 1.27 points out of a possible 2, a 63% average. Every student scored at least 1 point on these questions which is an improvement from the Midterm where 1 in 4 students got a 0. The students that received a 1 on this question simply stated that the domain was the "Time of Day" and the range was the "Temperature." They did not list the elements from the Table in set notation.

After 7 weeks, only those students that participated in the 'technology' lecture and took the Yellow Quiz #2 were asked some function questions on the Final exam as extra credit. The six students who participated in the 'technology' lecture that did not take the Final are not included in this analysis below. The complete list of questions is given in Appendix F. The analysis of the test scores from the Final are shown below in Figure 8 following with analysis of their responses.

Figure 8

Final Analysis

	Describe	Statement	Formula	Table	Domain (Table)	Range (Table)
Statistics	**Q1**	**Q2 Total**	**Q3 Total**	**Q4 Sub-Total**	**Q4c**	**Q4d**
Possible Points	2	8	4	4	2	2
Mean	1.00	1.67	1.78	3.67	0.89	0.89
Median	1	1	2	4	1	1
Mode	1	1	3	4	1	1
Standard Deviation	0.00	1.41	1.39	1.00	0.33	0.33
Minimum	1	0	0	1	0	0
Maximum	1	5	3	4	1	1
Count	9	9	9	9	9	9
Number of 0s	0	1	3	0	1	1
Number of 1's	9	4	0	1	8	8
Number of 2's	0	3	2	0	0	0
Number of 3's	-	0	4	0	-	-
Number of 4's	-	0	0	8	-	-
Number of 5's	-	1	-	-	-	-
Number of 6's	-	0	-	-	-	-
Number of 7's	-	0	-	-	-	-
Number of 8's	-	0	-	-	-	-

The first question on the Final was the exact same question that has been on all the other tests. It asked them to define a function and the data shows that the average score was 1.00 points out of a possible 2, a 50% average. This was the highest average score for this question and every student showed some knowledge of functions but no students received full credit

Question 2 asked to determine whether the statement: "An insurance company assigns to every customer an id number that matches the person's cell phone number" represented a function. The correct answer to this question depends on how the student defined the domain and range. If they used the set of customers as the domain and the set of possible cell phone numbers as the range, than this would not be a function because you cannot assume that everyone has a cell phone. If they switched the domain and range and stated that this was a function because each cell phone number can only go to one person, they earned full credit. The average score was only 1.67 points out of a possible 8, a 21% average. This is the lowest average score received since the Pre-Test on any question related to a statement. Most students identified the domain as the "i.d. number" and the range as the "cell phone number." However, according to the way the statement was worded, the i.d. number and the cell phone number are the same thing. Again, Alan stated that the domain and range was all real numbers even though this was a statement. Only one student, Joel, restricted the domain to only customers with cell phones and answered correctly.

Question 3 asked to determine whether the formula $h(x) = x^4 + x - 16$ gives a well-defined function for all real numbers. Since there was no division symbol or square roots involved in this formula every student should have answered this correctly even if they were not familiar with fourth powers because there were no 'bad' points. Disappointingly, the average score was 1.78 points out of a possible 4, a 44% average. This average score is slightly higher than the scores received on the first Quiz and the Midterm and is slightly lower than those from the second Quiz where the average score was 2.07 points. Three students claimed that this was not a function for different reasons. One student simply pointed to the x^4 in the equation, Joel; another student claimed that this could not be a function because 16 is in the domain, Alexandria; and the other student stated that this cannot be a function because "the answer could equal 0", Kristen. It appears that these students knew to look for 'bad' points, but they did not fully understand how to find inputs that don't result in real number outputs.

The fourth and final question about functions asked to determine whether a table represented a function. Similar to the Midterm and Quiz #2, this table has an element of the domain, "150", that did not correspond to any element of the range and thus cannot be a function. The average score for Question 4 was 3.67 points out of a possible 4, a 92% average! Only one student did not answer this question completely correctly. This is a significant improvement from the average scores on the Midterm and Quiz #2. It is clear that these students understood this representation of a function and what elements to look for when determining whether a table represents a function. The average score for identifying both the domain and range in Question 4 was 0.89 points out of a possible 2, a 44% average. Eight out of the nine students correctly identified the domain and range but did not use the proper notation. They simply said the domain was "horsepower" instead of listing the elements of the domain in set-notation. The only student to receive a '0' on this, Alan, stated that the domain was $\{x | x \text{ is R but } x \neq 150\}$ and the range was $\{y | y \neq 0\}$. He correctly stated that this table does not represent a function because the input of 150 horsepower didn't match up to any output, but could not properly identify the set of inputs and outputs for this correspondence.

CHAPTER 5

Results:

This case study was designed to compare the performance of students exposed to two different teaching methods, the 'traditional' lecture (based on textbook and whiteboard use) and the 'technology' lecture (based on computer visualizations and technology use). Participants reported to spend about 3 hours on their computer daily and several students claimed to be on their computers over 5 hours a day for various reasons.

We started with the hypothesis that a visual interactive lecture will significantly improve students' understanding of the function concept. The results show that while the improvement happened, it was not as significant as we had hoped for. However, we found that this study supports the slightly different hypothesis that students respond more positively, learn more, and perform better on tests when taught functions using 'technology' after being taught using a 'traditional' method. Fourteen students took Quiz #1 and the Yellow Quiz #2 and both tests were graded consistently. We used a paired one-tailed t test with a confidence level of 90% to evaluate whether the students' improvement was statistically significant after the 'technology' lecture. See "*An Introduction to Mathematical Statistics and its Applications*" [4] for the technique.

H_0 = the mean difference between teaching students functions the 'traditional' way and 'technology' way is zero.

H_A = the mean difference between teaching students functions the 'technology' way is greater than the 'traditional' way.

Whenever the p-value was less than 0.10, we can say with at least 90% confidence that students learned more after the 'technology' lecture.

We started with calculating the p-value by first finding the difference between the test scores from the Yellow Quiz #2 to the test scores on Quiz #1. Then we calculated the mean, $\bar{d}$, and standard deviations, s_d, of the differences and used this information along with the fact that 14 students took both tests to calculate the t ratio of the mean difference. The formula we used for calculating t is found in Figure 9 below. Then we used the Microsoft Excel function "TDIST" to calculate the p-value with 13 degrees of freedom. Also, the upper percentiles of the Student t distribution from the Mathematical Statistics book [4] are shown in Figure 10 below and were used to calculate the confidence intervals.

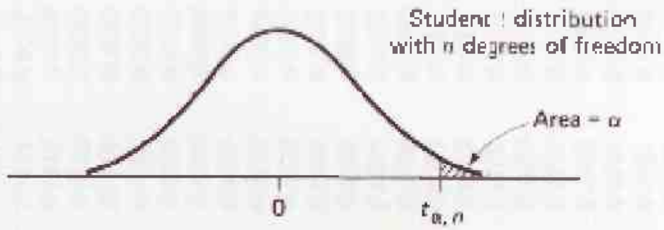
In order to evaluate the hypothesis, we compared both the total score received from each student on Quiz #1 to the score received on Quiz #2, as well as the scores received on each type of question individually to determine whether certain representations of functions caused more issues than others. In testing the hypothesis, we only used test scores from students who attended both the 'traditional' and 'technology' lectures and took both tests being compared. The summary of the test scores and the hypothesis testing results are shown in Figure 11 below.

Figure 9

t ratio of the mean difference: $t = \frac{\bar{d}}{s_d/\sqrt{n}}$

Figure 10

TABLE A.2 UPPER PERCENTILES OF STUDENT t DISTRIBUTIONS



df	α = 0.20	0.15	0.10	0.05	0.025	0.01	0.005
1	1.376	1.963	3.078	6.3138	12.706	31.821	63.657
2	1.061	1.386	1.886	2.9200	4.3027	6.965	9.9248
3	0.978	1.250	1.638	2.3534	3.1825	4.541	5.8409
4	0.941	1.190	1.533	2.1318	2.7764	3.747	4.6041
5	0.920	1.156	1.476	2.0150	2.5706	3.365	4.0321
6	0.906	1.134	1.440	1.9432	2.4469	3.143	3.7074
7	0.896	1.119	1.415	1.8946	2.3646	2.998	3.4995
8	0.889	1.108	1.397	1.8595	2.3060	2.896	3.3554
9	0.883	1.100	1.383	1.8331	2.2622	2.821	3.2498
10	0.879	1.093	1.372	1.8125	2.2281	2.764	3.1693
11	0.876	1.088	1.363	1.7959	2.2010	2.718	3.1058
12	0.873	1.083	1.356	1.7823	2.1788	2.681	3.0545
13	0.870	1.079	1.350	1.7709	2.1604	2.650	3.0123
14	0.868	1.076	1.345	1.7613	2.1448	2.624	2.9768
15	0.866	1.074	1.341	1.7530	2.1315	2.602	2.9467
16	0.865	1.071	1.337	1.7459	2.1199	2.583	2.9208
17	0.863	1.069	1.333	1.7396	2.1098	2.567	2.8982
18	0.862	1.067	1.330	1.7341	2.1009	2.552	2.8784
19	0.861	1.066	1.328	1.7291	2.0930	2.539	2.8609
20	0.860	1.064	1.325	1.7247	2.0860	2.528	2.8453
21	0.859	1.063	1.323	1.7207	2.0796	2.518	2.8314
22	0.858	1.061	1.321	1.7171	2.0739	2.508	2.8188
23	0.858	1.060	1.319	1.7139	2.0687	2.500	2.8073
24	0.857	1.059	1.318	1.7109	2.0639	2.492	2.7969
25	0.856	1.058	1.316	1.7081	2.0595	2.485	2.7874
26	0.856	1.058	1.315	1.7056	2.0555	2.479	2.7787
27	0.855	1.057	1.314	1.7033	2.0518	2.473	2.7707
28	0.855	1.056	1.313	1.7011	2.0484	2.467	2.7633
29	0.854	1.055	1.311	1.6991	2.0452	2.462	2.7564
30	0.854	1.055	1.310	1.6973	2.0423	2.457	2.7500
31	0.8535	1.0541	1.3095	1.6955	2.0395	2.453	2.7441
32	0.8531	1.0536	1.3086	1.6939	2.0370	2.449	2.7385
33	0.8527	1.0531	1.3078	1.6924	2.0345	2.445	2.7333
34	0.8524	1.0526	1.3070	1.6909	2.0323	2.441	2.7284

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Figure 11

Quiz #1 vs. Quiz #2 (Yellow)

Name	Quiz #	Q1 Func.	Q2a1 Y/N	Q2a2 Why	Q2a3 Domain	Q2a4 Range	Q2b1 Y/N	Q2b2 Why	Q2b3 Domain	Q2b4 Range	Q2ab Total	Q2c1 Y/N	Q2c2 Why	Q2c Total	Q2d1 Y/N	Q2d2 Why	Q2d Total	Q2e1 Y/N	Q2e2 Why	Q2e Total	Total Score
Alan	1	1	0	0	0	0	2	1	0	0	3	0	0	0	2	0	2	2	2	4	10
	2	1	0	0	0	0	2	1	1	1	5	2	0	2	0	1	1	0	1	1	10
Difference		0	0	0	0	0	0	0	1	1	2	2	0	2	-2	1	-1	-2	-1	-3	0
Alexandria	1	0	0	1	0	1	2	2	1	1	8	0	0	0	0	0	0	2	0	2	10
	2	1	0	1	1	2	2	1	1	1	9	2	1	3	0	0	0	2	2	4	17
Difference		1	0	0	1	1	0	-1	0	0	1	2	1	3	0	0	0	0	2	2	7
Andrew	1	1	0	0	0	1	2	0	1	1	5	2	0	2	2	0	2	0	0	0	10
	2	0	2	2	1	1	2	1	1	1	11	2	0	2	2	2	4	2	2	4	21
Difference		-1	2	2	1	0	0	1	0	0	6	0	0	0	0	2	2	2	2	4	11
Aylana	1	0	2	0	0	0	2	0	1	1	6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	6
	2	0	0	0	1	1	2	1	1	1	7	0	0	0	0	0	0	2	2	4	11
Difference		0	-2	0	1	1	0	1	0	0	1	0	0	0	0	0	0	2	2	4	5
Brittany	1	0	0	1	1	1	2	2	1	1	9	0	0	0	0	0	0	2	1	3	12
	2	1	2	2	2	2	0	0	1	1	10	0	0	0	2	2	4	2	2	4	19
Difference		1	2	1	1	1	-2	-2	0	0	1	0	0	0	2	2	4	0	1	1	7
Candace	1	0	0	1	0	1	2	2	1	1	8	2	0	2	2	0	2	2	2	4	16
	2	1	0	1	1	1	0	0	1	1	5	0	0	0	2	2	4	0	0	0	10
Difference		1	0	0	1	0	-2	-2	0	0	-3	-2	0	-2	0	2	2	-2	-2	-4	-6
Elsa	1	0	2	[illegible]	0	0	[illegible]	[illegible]	[illegible]	1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	10
	2	1	0	1	1	2	0	0	1	1	6	0	1	1	0	1	1	2	2	4	13
Difference		1	-2	1	0	1	-2	-1	0	0	-3	0	1	1	0	0	0	2	2	4	3
Joel	1	1	2	[illegible]	0	0	[illegible]	[illegible]	1	1	10	2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	2	4	21
	2	1	0	1	1	1	0	0	1	1	5	2	1	3	0	1	1	0	1	1	11
Difference		0	-2	-1	1	1	-2	-2	0	0	-5	0	1	1	-2	-1	-3	-2	-1	-3	-10
Karl	1	1	2	2	0	0	2	2	1	1	10	0	0	0	2	0	2	2	2	4	17
	2	1	0	1	1	1	2	1	1	1	8	2	2	4	0	1	1	0	0	0	14
Difference		0	-2	-1	1	1	0	-1	0	0	-2	2	2	4	-2	1	-1	-2	-2	-4	-3
Kristen	1	0	[illegible]	[illegible]	0	0	[illegible]	[illegible]	[illegible]	1	4	0	0	[illegible]	[illegible]	1	3	2	2	4	11
	2	1	0	0	1	1	0	1	1	1	5	2	2	4	2	0	2	2	2	4	16
Difference		1	-2	0	1	1	0	1	0	0	1	2	2	4	0	-1	-1	0	0	0	5
Lauren	1	2	0	1	0	0	[illegible]	[illegible]	0	0	5	2	2	[illegible]	[illegible]	[illegible]	[illegible]	2	2	4	16
	2	1	0	1	1	1	2	1	1	1	8	2	2	4	0	1	1	2	2	4	18
Difference		-1	0	0	1	1	0	-1	1	1	3	0	0	0	0	0	0	0	0	0	2
Matt	1	0	0	[illegible]	[illegible]	[illegible]	2	2	1	1	9	2	1	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	15
	2	1	0	1	1	1	2	2	1	1	9	2	1	3	0	1	1	2	2	4	18
Difference		1	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	-2	2	2	4	3
Maura	1	1	0	1	0	0	2	2	1	1	7	0	0	0	0	1	1	2	2	4	13
	2	2	2	2	2	2	2	2	1	1	14	0	1	1	0	1	1	2	2	4	22
Difference		1	2	1	2	2	0	0	0	0	7	0	1	1	0	0	0	0	0	0	9
Ryan	1	0	[illegible]	[illegible]	0	1	2	1	1	1	9	0	0	0	0	0	0	2	0	2	11
	2	0	0	1	1	1	2	0	1	1	7	2	1	3	0	1	1	0	0	0	11
Difference		0	-2	0	1	0	0	-1	0	0	-2	2	1	3	0	1	1	-2	0	-2	0

Statistics for Quiz #1	Describe Q1	Statement Q2a1	Q2a2	Q2a3	Q2a4	Statement Q2b1	Q2b2	Q2b3	Q2b4	Statement Total	Formula Q2c1	Q2c2	Formula Total	Graph Q2d1	Q2d2	Graph Total	Table Q2e1	Q2e2	Table Total	Total Score
Possible Points	2	2	2	2	2	2	2	2	2	16	2	2	4	2	2	4	2	2	4	30
Mean	0.50	0.86	0.79	0.21	0.50	1.86	1.36	0.86	0.86	7.29	0.71	0.21	0.93	1.00	0.50	1.50	1.43	1.07	2.50	12.71
Standard Deviation	0.65	1.03	0.70	0.43	0.52	0.53	0.84	0.36	0.36	2.30	0.99	0.58	1.38	1.04	0.65	1.29	0.94	1.00	1.79	3.87

Statistics for Quiz #2	Describe Q1	Statement Q2a1	Q2a2	Q2a3	Q2a4	Statement Q2b1	Q2b2	Q2b3	Q2b4	Statement Total	Formula Q2c1	Q2c2	Formula Total	Graph Q2d1	Q2d2	Graph Total	Table Q2e1	Q2e2	Table Total	Total Score
Possible Points	2	2	2	2	2	2	2	2	2	16	2	2	4	2	2	4	2	2	4	30
Mean	0.86	0.43	1.00	1.07	1.21	1.29	0.79	1.00	1.00	7.79	1.29	0.86	2.14	0.57	1.00	1.57	1.29	1.43	2.71	15.07
Standard Deviation	0.53	0.85	0.68	0.47	0.58	0.99	0.70	0.00	0.00	2.67	0.99	0.77	1.51	0.94	0.68	1.40	0.99	0.85	1.82	4.18

Statistics for Differences	Describe Q1	Statement Q2a1	Q2a2	Q2a3	Q2a4	Statement Q2b1	Q2b2	Q2b3	Q2b4	Statement Total	Formula Q2c1	Q2c2	Formula Total	Graph Q2d1	Q2d2	Graph Total	Table Q2e1	Q2e2	Table Total	Total Score
Mean	0.36	-0.43	0.21	0.86	0.71	-0.57	-0.57	0.14	0.14	0.50	0.57	0.64	1.21	-0.43	0.50	0.07	-0.14	0.36	0.21	2.36
Standard Deviation	0.74	1.60	0.80	0.53	0.61	0.94	1.09	0.36	0.36	3.39	1.22	0.74	1.76	1.16	1.02	1.77	1.66	1.50	3.07	5.81
Count	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14
t Ratio	1.794	-1.000	1.000	6.000	4.372	-2.280	-1.963	1.472	1.472	0.552	1.749	3.229	2.579	-1.385	1.836	0.151	-0.322	0.891	0.261	1.517
Degrees of Freedom	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13
p-value	0.0481	0.8322	0.1678	0.0000	0.0004	0.9800	0.9643	0.0824	0.0824	0.2953	0.0519	0.0033	0.0115	0.9053	0.0447	0.4413	0.6239	0.1945	0.3990	0.0766
	Reject H_o in favor of H_a									Accept H_o			Reject H_o in favor of H_a			Accept H_o			Accept H_o	Reject H_o in favor of H_a

Hypothesis Testing

H_0: The mean difference between teaching students functions the 'traditional' way and 'technology' way is zero

H_A: The mean difference between teaching students functions the 'technology' way is greater than the 'traditional' way

Reject H_0 in favor of H_A if p-value < 0.10

The calculated *p*-value of 0.0766 from comparing the Total Score between the first and second quizzes was less than 0.1. Therefore with 90% confidence we can say that students responded, learned, and performed better on tests after being taught functions using the technology based lecture. However, even though their scores improved significantly, their average score was only 15.07 points out 30 possible points, a 50% average, hence still very low. Only five students received a passing grade and the highest grade was 22 out of 30, a 73%. Three students actually performed worse after the 'technology' lecture than they did after the 'traditional' lecture. This implies that even though the average student improved as a result of the 'technology' lecture, something is still missing from the way functions are taught that is necessary to build a complete concept image of functions. Therefore our method, while successful, did not improve the students' understanding to the mastery level.

Interestingly, when comparing the test results for questions dealing with the different representations of functions, the students only significantly improved on the definition and formula representations of functions. Even though the questions about whether or not statements represented functions were very similar on the two quizzes, minimal score gains were recorded. This may have been caused by the fact that these students appear to struggle with word problems in general. A possible reason for not seeing improvement on the question related to graphs may be due to the different 'looking' graphs were used on each quiz. Based on the students' responses, many struggled with the graph from Quiz #2 since it was broken up into 3 pieces. On both quizzes, the students struggled with the 'holes' in the graph and were not sure how to interpret them. Both correspondences defined by tables on Quiz #1 and Quiz #2 were not functions, but for different reasons. The table on the first Quiz was not a function because one input went to two different outputs and the table on the second Quiz was not a function because one of the inputs did not go to any output. It is possible that the students performed poorly on these two questions for different reasons. And if the questions were similar, they may have shown some improvement.

According to the above data, the average improvement was only 2.36 points which is relatively small when compared to the perfect score being 30 points and the average score for these students on Quiz #1 was only 12.71 points. To calculate the 90% confidence interval for the true mean difference we used the formula shown below in Figure 12 from the Mathematical Statistics book [4] to come up with $2.36 \pm 2.097 = (0.26, 4.45)$. This indicates that with 90% confidence we can conclude that the mean increased after the 'technology' lecture significantly.

Figure 12

$\bar{d} \pm t_{\alpha,n-1}\frac{s_d}{\sqrt{n}}$

or,

equivalently $(\bar{d} - t_{\alpha,n-1}\frac{s_d}{\sqrt{n}}, \bar{d} + t_{\alpha,n-1}\frac{s_d}{\sqrt{n}})$ where α represents the $100(1-\alpha)\%$ confidence level.

In addition to asking questions about the components of functions, we also asked qualitative questions on the Yellow Quiz #2 to determine whether or not these students preferred the 'technology' lecture over the 'traditional' ones. We asked the students whether or not they took notes during the 'technology' lecture and/or any of the previous 'traditional' lectures and which ones they preferred. We also asked them if they felt more comfortable and if they felt like they learned more during the 'technology' lecture than they did during the 'traditional' lectures. In order to quantitatively analyze their responses, we assigned to each response an integer value from -1 to +1 where +1 indicates the student preferred the 'technology' lecture and -1 indicates the student preferred the 'traditional' lectures. A score of 0 meant that the student did not have a preference one way or another. Another question was asked to determine each student's familiarity with computers by determining how much time they spent on them daily. Usually we assume that the more time the student spent on the computer, the more familiar they were with them and use computers as an important means of communication. Figure 13 below summarizes the students' responses and also gives a Total Score to evaluate their preference of using computers in the classroom. The maximum possible score for this was 8 points.

Figure 13

Preferred Technology Lecture?

Name	Last Lecture Notes	Previous Lecture Notes	More Comfortable in Last or Previous	Own or Prepared Notes Better	Know more after PP Lecture	Total
Alan	Yes	Yes	Yes	Prepared	Yes	3
Alexandria	Yes	Yes	Yes	Prepared	Yes	3
Andrew	Yes	Yes	Last	Both	Yes	2
Atiana	Yes	Yes	Somewhat	Depends	Somewhat	0
Brittany	Yes	Sometimes	Last	Prepared	Yes	4
Candace	Yes	Yes	Last	My Own	Yes	1
Elsa	Yes	Yes	Last	Prepared	Yes	3
Joel	Yes	Yes	Previous	My Own	Yes	-1
Karl	Yes	No	Both	Prepared	Yes	4
Kristen	Yes	Yes	Last	Prepared	Yes	3
Lauren	Yes	Yes	Last	Prepared	Sure	2
Matt	Yes	Yes	Yes	Prepared	Blank	2
Maura	No	Yes	No	My Own	No	-5
Ryan	No	Yes	Yes	Prepared	Yes	1
Total	10	-11	8	6	9	22

Scoring Scheme					
	Yes = 1	Yes = -1	Yes/Last = 1	Prepared = 1	Yes = 1
		Sometimes = 0	Both/Somewhat = 0	Both Depends = 0	Somewhat Sure Blank = 0
	No = -1	No = 1	No/Previous = -1	My Own = -1	No = -1

Time on Computer

Name	Hours a Day	Score	Total Score
Alan	1.5 hours	1	4
Alexandria	way too much time	3	6
Andrew	2 - 3 hrs	2	4
Atiana	2 - 3 hrs	2	2
Brittany	a lot	3	7
Candace	1 - 2 hrs	1	2
Elsa	2 - 4 hrs	2	5
Joel	1 hour	0	-1
Karl	2 hrs	1	5
Kristen	2 hrs	1	4
Lauren	a lot	3	5
Matt	4 hrs	2	4
Maura	1 hour	0	-5
Ryan	5 hrs	3	4
Total		24	46

Scoring Scheme	
	1 Hour or Less = 0
	1.1 to 2 Hours = 1
	2.1 to 4 hours = 2
	4.1 or more hours = 3

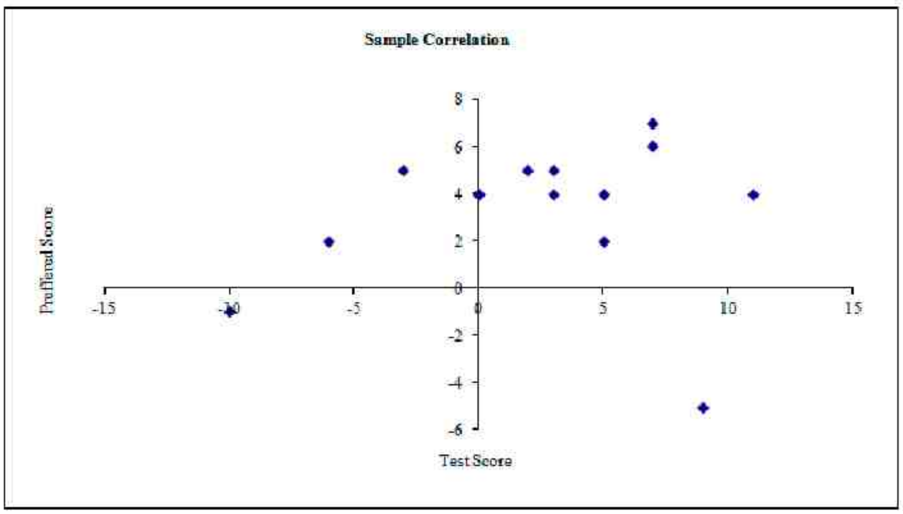
All but two students preferred the ‘technology’ lecture over the ‘traditional’ ones. One of those students, Maura, actually supported the ‘traditional’ lecture in every category and also spent very little time on the computer. Even though most students preferred the ‘technology’ lecture, was there a correlation with higher test scores? To determine whether their preference of the ‘technology’ lecture correlated to their higher test scores, we took their preference scores from above and the difference between their test scores from Quiz #1 and the Yellow Quiz #2 and calculated the Sample Correlation according to the formula in Figure 14 below. Based on the results shown in Figure 14, the Correlation Coefficient is 0.136 meaning that there is almost no correlation between each student’s improvement and their preference of the ‘technology’ lecture.

Figure 14

Sample Correlation: $r_{xy} = \frac{\sum_{i=1}^{i=n}(x_i - \mu_x)(y_i - \mu_y)}{(n-1)\sigma_x\sigma_y} = \frac{\sum_{i=1}^{i=n} x_i y_i - n\mu_x\mu_y}{(n-1)\sigma_x\sigma_y}$

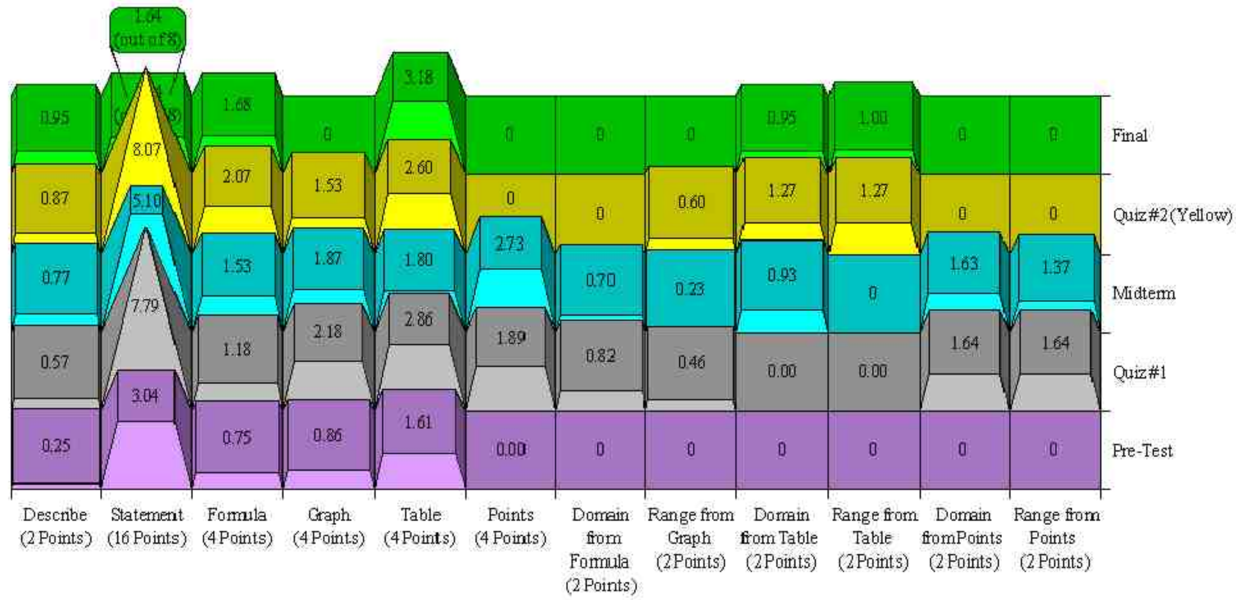
Average score for each component

Test	Name	Improvement	Preffered Score	Mean Test Score	S.D. Test Score	Mean Pref. Score	S.D. Pref. Score	Count	Sample Correlation
Quiz #1 vs. Quiz #2 Difference	Alan	0	4	2.36	5.813	3.29	3.074	14	0.136
	Alexandria	7	6						
	Andrew	11	4						
	Acylana	5	2						
	Brittany	7	7						
	Candace	-6	2						
	Elsa	3	5						
	Joel	-10	-1						
	Karl	-3	5						
	Kristen	5	4						
	Lauren	2	5						
	Matt	3	4						
	Maura	9	-5						
	Ryan	0	4						



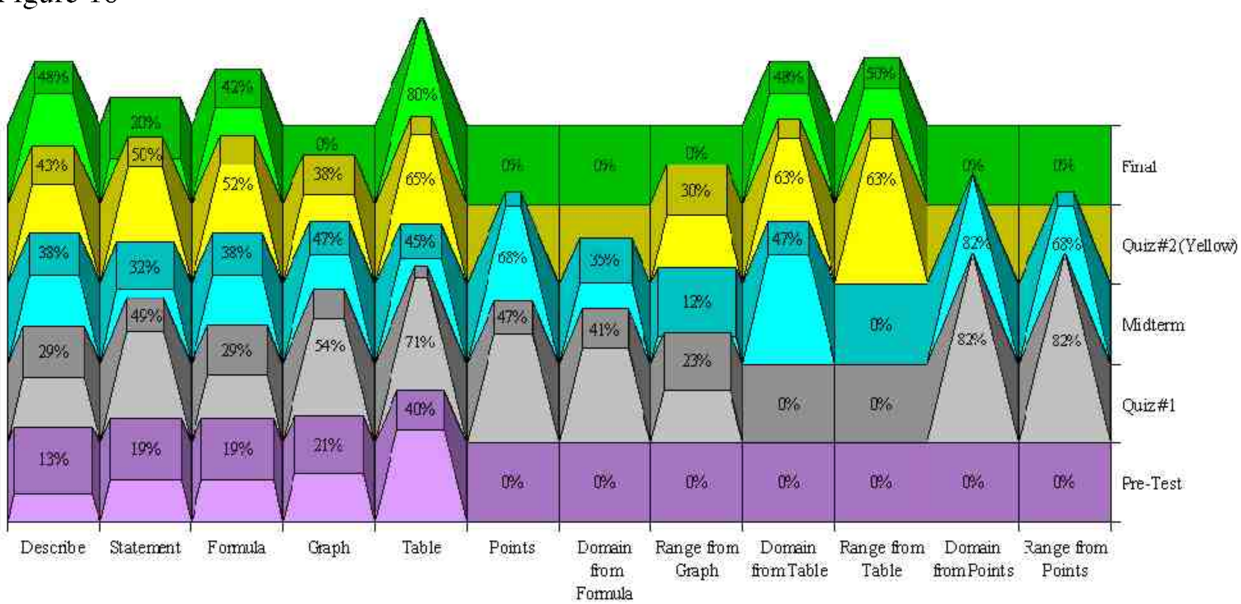
Now we will analyze the scores of all consecutive tests to evaluate the progression of the students' understanding of the function concept. Figure 15 below, shows the average test score for each function question from the Pre-Test, Quiz #1, Midterm, Quiz #2, and the Final for all students who took these tests.

Figure 15



We rescale these raw scores into percent correct scores received for each function question from the Pre-Test, Quiz #1, Midterm, Quiz #2, and the Final for all students who took these tests. This is shown in Figure 16 below.

Figure 16



Several observations can be made here. First, every time the students were asked to "Explain what a function is", their test scores improved. However when you look at the average grade received for this question, it peaked at 48%. This means that the group improved, but never understood the complete definition of a function. The constant improvement on this question may be a result of asking the exact same question on each test and then reviewing the definition of a function after handing their tests back and going over the answers. Similarly, when asked whether a formula represented a function, the group steadily improved on each test (except for the Final) but they never showed full understanding. The hypothesis testing done earlier showed that statistically speaking, the students performed better on these two representations of functions after the 'technology' lecture than they did after the 'traditional' lecture.

Interestingly, the students performed their best on the statement questions immediately following the lectures (Quiz #1 and Quiz #2) and did not perform well on either of the tests after going through several weeks of non-function related algebra activities (Midterm and Final). Their scores never averaged higher than 50% correct for this type of question perhaps because they struggle with word problems. The group performed the best, on average, when asked whether a table represented a function. Only on the Pre-Test and Midterm did the students average a failing grade. On the Final they averaged 80%. This was expected since the table representation is the clearest representation of the function because it separates the domain and range well and the correspondence connecting the elements is also very clear. If the students would have translated the other representations into tables, they would likely have been able to perform better on those questions. This should be researched further.

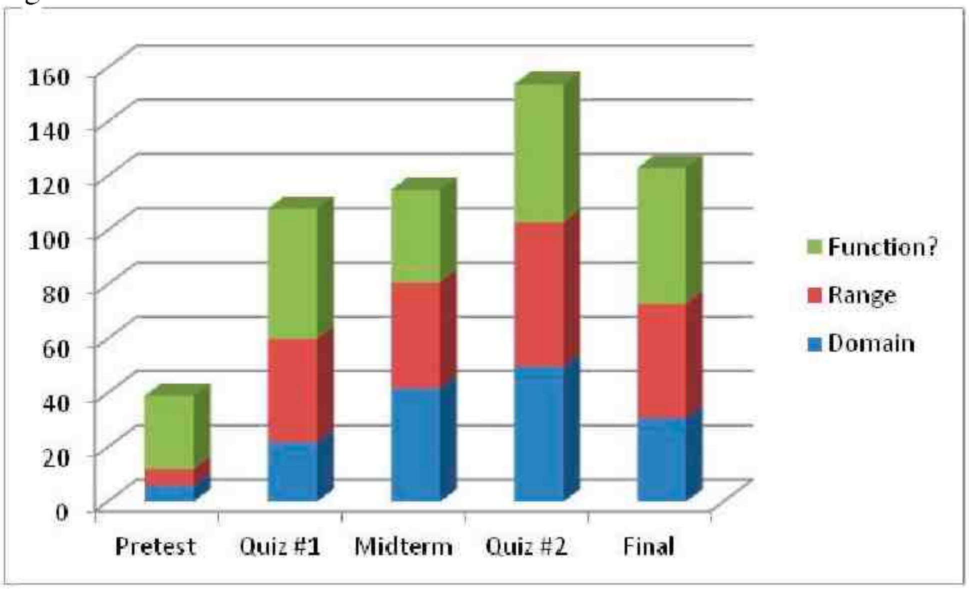
The question about whether a graph represented a function was not asked on the Final, but looking at the test scores received on the other tests, it is clear that after the 'traditional' lecture, there was no improvement. This may be due to the questions getting slightly harder on each test. On Quiz #1, the graph had one point removed, but still passed the VLT. Seven of the twenty-eight students who took this test correctly made this observation. On the Midterm, there was a point added below the solid curve that failed the VLT. Data shows that the group knew that the VLT was a good tool to use to tell whether graphs represent a function, but only about 25% of them knew how to use it correctly. It is also possible that the students struggled with the notation and not the function concept because they did not understand the meaning of the open and closed points on the graph.

Questions about whether a series of pairs represented a function were only asked on the Quiz #1 and the Midterm. Surprisingly, the group performed better on the Midterm where they averaged 2.73 points out of a possible 4, a 68% average, than they did on Quiz #1 even though we had not discussed the function topic for several weeks. This unexpected increase in understanding may be a result of the students studying the material more in preparation of the Midterm and possibly even reviewing the answers to the first Quiz. Without having asked this type of question after the 'technology' lecture, it is hard to predict whether or not these scores would have improved.

The domain and range questions displayed in Figures 15 and 16 above show which tests asked what questions and the average scores received on these questions. If the data showed 0% average, then that type of question was not asked on that test. These figures show that the group struggled to properly identify the domain and the range each time. They struggled more when identifying the range.

The interesting subset of the above data contains scores of the nine students who took every test given on functions (Pre-Test, Quiz #1, Midterm, Quiz#2 Yellow, and Final). This helps to track this group's progress throughout the study. Due to the fact that each test had different numbers of function questions, we standardized their scores by finding the percentage correct for each of the three function related questions on each test, multiplied that by 100 and then added them together for a maximum possible score of 300 points. A summary of this data is shown in Figure 17.

Figure 17



This histogram displays the trend that the total score increased on each test except for the Final (but they scored higher than on the Midterm). Note that the Midterm and Final were high stakes exams that contained many other algebra questions besides functions. Hence lower scores are not surprising there. This data coincides with the hypothesis testing earlier where there was a statistically significant increase in the students' understanding of the function concept after the 'technology' lecture. Also, looking at each component individually, you can see how limited their cognitive roots were at the start of class. The biggest gains these students had in test scores came from understanding the domain and range. However, looking at the detailed data by student shown in Figure 18, it is clear that even at the peak of their test scores, they did not fully understand the concept of a function.

Figure 18

Average score for each component

Test	Name	Domain	Possible Points	% Correct	Range	Possible Points	% Correct	Function?	Possible Points	% Correct	Total Score
Pretest	Alan	0	4	0%	0	4	0%	6	22	27%	27
	Alexandria	0	4	0%	0	4	0%	4	22	18%	13
	Andrew	0	4	0%	0	4	0%	8	22	36%	36
	Atilana	0	4	0%	0	4	0%	2	22	9%	9
	Brittany	0	4	0%	0	4	0%	6	22	27%	27
	Joel	0	4	0%	0	4	0%	6	22	27%	27
	Karl	0	4	0%	0	4	0%	7	22	32%	32
	Kristen	0	4	0%	0	4	0%	5	22	23%	23
	Lauren	2	4	50%	2	4	50%	10	22	45%	145
	Total	2	36	6%	2	36	6%	54	198	27%	33
Quiz #1	Alan	0	4	0%	0	4	0%	10	22	45%	45
	Alexandria	1	4	25%	2	4	50%	7	22	32%	107
	Andrew	1	4	25%	2	4	50%	7	22	32%	107
	Atilana	1	4	25%	1	4	25%	4	22	18%	63
	Brittany	2	4	50%	2	4	50%	8	22	36%	136
	Joel	1	4	25%	1	4	25%	19	22	86%	136
	Karl	1	4	25%	1	4	25%	15	22	68%	113
	Kristen	1	4	25%	1	4	25%	9	22	41%	91
	Lauren	0	4	0%	0	4	0%	16	22	73%	73
	Total	8	36	22%	10	36	28%	95	198	48%	93
Midterm	Alan	1	12	8%	0	10	0%	12	34	35%	44
	Alexandria	6	12	50%	5	10	50%	11	34	32%	132
	Andrew	8	12	67%	6	10	60%	10	34	29%	156
	Atilana	4	12	33%	3	10	30%	12	34	35%	99
	Brittany	4	12	33%	4	10	40%	13	34	38%	111
	Joel	5	12	42%	2	10	20%	12	34	35%	97
	Karl	7	12	58%	6	10	60%	9	34	26%	145
	Kristen	5	12	42%	4	10	40%	7	34	21%	102
	Lauren	5	12	42%	5	10	50%	18	34	53%	145
	Total	45	108	42%	35	90	39%	104	305	34%	115
Quiz #2	Alan	1	4	25%	1	4	25%	8	22	36%	86
	Alexandria	2	4	50%	3	4	75%	12	22	55%	130
	Andrew	2	4	50%	2	4	50%	17	22	77%	177
	Atilana	2	4	50%	2	4	50%	7	22	32%	132
	Brittany	3	4	75%	3	4	75%	13	22	59%	209
	Joel	2	4	50%	2	4	50%	7	22	32%	132
	Karl	2	4	50%	2	4	50%	10	22	45%	145
	Kristen	2	4	50%	2	4	50%	12	22	55%	155
	Lauren	2	4	50%	2	4	50%	14	22	64%	164
	Total	18	36	50%	19	36	53%	100	198	51%	153
Final	Alan	0	4	0%	0	4	0%	8	14	57%	57
	Alexandria	2	4	50%	2	4	50%	5	14	36%	136
	Andrew	1	4	25%	2	4	50%	7	14	50%	125
	Atilana	1	4	25%	1	4	25%	9	14	64%	114
	Brittany	1	4	25%	2	4	50%	5	14	36%	111
	Joel	2	4	50%	2	4	50%	8	14	57%	157
	Karl	2	4	50%	2	4	50%	8	14	57%	157
	Kristen	1	4	25%	2	4	50%	5	14	36%	111
	Lauren	1	4	25%	2	4	50%	8	14	57%	132
	Total	11	36	31%	16	36	42%	63	125	50%	122
										Total Possible	300

Another way to look at our data is to reorganize the students' test scores to track their individual progress from the Pre-Test to the Final. We calculated the average improvement from Quiz #1 and the Midterm against the Pre-Test (tests after the 'traditional' lectures) and also the average improvement from Quiz #2 Yellow and the Final against the Pre-Test (tests after the 'technology' lecture). This data is presented in Figure 19.

Figure 19

Average score for each component

Name	Test	Domain	Possible Points	% Correct	Range	Possible Points	% Correct	Function?	Possible Points	% Correct	Total Score	Average Improvement from Pre-Test	Average Improvement after Technology Lecture
Alan	Pretest	0	4	0%	0	4	0%	6	22	27%	27		
	Quiz #1	0	4	0%	0	4	0%	10	22	45%	45	17	
	Midterm	1	12	8%	0	10	0%	12	34	35%	44		27
	Quiz #2	1	4	25%	1	4	25%	8	22	36%	86	44	
	Final	0	4	0%	0	4	0%	8	14	57%	57		
Alexandria	Pretest	0	4	0%	0	4	0%	4	22	18%	18		
	Quiz #1	1	4	25%	2	4	50%	7	22	32%	107	101	
	Midterm	6	12	50%	5	10	50%	11	34	32%	132		38
	Quiz #2	2	4	50%	3	4	75%	12	22	55%	180	139	
	Final	2	4	50%	2	4	50%	5	14	36%	136		
Andrew	Pretest	0	4	0%	0	4	0%	8	22	36%	36		
	Quiz #1	1	4	25%	2	4	50%	7	22	32%	107	95	
	Midterm	8	12	67%	6	10	60%	10	34	29%	157		20
	Quiz #2	2	4	50%	2	4	50%	17	22	77%	177	115	
	Final	1	4	25%	2	4	50%	7	14	50%	125		
Atiyana	Pretest	0	4	0%	0	4	0%	2	22	9%	9		
	Quiz #1	1	4	25%	1	4	25%	4	22	18%	68	74	
	Midterm	4	12	33%	3	10	30%	12	34	35%	99		40
	Quiz #2	2	4	50%	2	4	50%	7	22	32%	13[illegible]	114	
	Final	1	4	25%	1	4	25%	9	14	64%	1[illegible]4		
Brittany	Pretest	0	4	0%	0	4	0%	6	22	27%	27		
	Quiz #1	2	4	50%	2	4	50%	8	22	36%	136	97	
	Midterm	4	12	33%	4	10	40%	13	34	38%	112		36
	Quiz #2	3	4	75%	3	4	75%	13	22	59%	209	133	
	Final	1	4	25%	2	4	50%	5	14	36%	111		
Joel	Pretest	0	4	0%	0	4	0%	6	21	27%	27		
	Quiz #1	1	4	25%	1	4	25%	19	22	86%	136	89	
	Midterm	5	12	42%	2	10	20%	12	34	35%	97		28
	Quiz #2	2	4	50%	2	4	50%	7	22	32%	132	117	
	Final	2	4	50%	2	4	50%	8	14	57%	157		
Karl	Pretest	0	4	0%	0	4	0%	7	21	32%	32		
	Quiz #1	1	4	25%	1	4	25%	15	22	68%	118	100	
	Midterm	7	12	58%	6	10	60%	9	34	26%	145		20
	Quiz #2	2	4	50%	2	4	50%	10	21	45%	145	119	
	Final	2	4	50%	2	4	50%	8	14	57%	157		
Kristen	Pretest	0	4	0%	0	4	0%	5	22	23%	23		
	Quiz #1	1	4	25%	1	4	25%	9	22	41%	91	74	
	Midterm	5	12	42%	4	10	40%	7	34	21%	102		36
	Quiz #2	2	4	50%	2	4	50%	12	22	55%	156	113	
	Final	1	4	25%	2	4	50%	5	14	36%	111		
Lauren	Pretest	2	4	50%	2	4	50%	10	22	45%	145		
	Quiz #1	0	4	0%	0	4	0%	16	22	73%	73	-37	
	Midterm	5	12	42%	5	10	50%	18	34	53%	145		39
	Quiz #2	2	4	50%	2	4	50%	14	21	64%	164	2	
	Final	1	4	25%	2	4	50%	8	14	57%	132		
									Total Possible		300		Overall Average = 32

The data shows every test score for every participant improved after the ‘technology’ lecture with the average student improving by 32 points (or about 10%). It is interesting that one student, Lauren, actually scored lower on the first Quiz than she did on the Pre-Test. However, her other test scores were similar to her score on the Pre-Test implying that her cognitive roots did not change after either the ‘traditional’ or the ‘technology’ lectures. Note that this data is compatible with the earlier evidence that the group did understand the function concept better after the ‘technology’ lecture.

Lastly, there were five students who performed well on the Midterm and also attended the 'technology' lecture. Their Quiz #2, the Blue version, contained more challenging questions. Figure 20 compares their scores from this second Quiz to their scores from similar questions on the Midterm.

Figure 20

Midterm vs. Quiz #2 (Blue)

Name	Test	Describe	Statement Why	Statement Domain	Statement Range	Statement Total	Formula Y/N	Formula Why	Formula Total	Domain from Formula	Graph How Make?	Statement How Make?	Total Score
Cristina	Mid	1	0	1	1	2	2	1	3	0	0	2	8
	Quiz #2	1	2	1	1	4	0	0	0	1	1	2	9
Difference		0	2	0	0	2	-2	-1	-3	1	1	0	1
Darren	Mid	1	1	0	0	1	2	1	3	1	1	0	7
	Quiz #2	2	2	2	1	5	0	0	0	1	1	1	10
Difference		1	1	2	1	4	-2	-1	-3	0	0	1	3
Francesca	Mid	1	0	1	1	2	2	0	2	1	0	0	6
	Quiz #2	1	2	1	1	4	2	0	2	1	1	2	11
Difference		0	2	0	0	2	0	0	0	0	1	2	5
Georgina	Mid	1	0	1	0	1	2	1	3	2	1	1	9
	Quiz #2	1	0	1	1	2	2	0	2	1	1	2	9
Difference		0	0	0	1	1	0	-1	-1	-1	0	1	0
Paige	Mid	1	0	1	0	1	2	1	3	2	1	2	10
	Quiz #2	1	2	2	2	6	0	0	0	1	0	2	10
Difference		0	2	1	2	5	-2	-1	-3	-1	-1	0	0

Statistics for Midterm	Describe	Statement Why	Statement Domain	Statement Range	Statement Total	Formula Y/N	Formula Why	Formula Total	Domain from Formula	Graph How Make?	Statement How Make?	Total Score
Possible Points	2	2	2	2	6	2	2	4	2	2	2	13
Mean	1.00	0.20	0.80	0.40	1.40	2.00	0.80	2.80	1.20	0.60	1.00	8.00
Standard Deviation	0.00	0.45	0.45	0.55	0.55	0.00	0.45	0.45	0.84	0.55	1.00	1.58

Statistics for Quiz #2	Describe	Statement Why	Statement Domain	Statement Range	Statement Total	Formula Y/N	Formula Why	Formula Total	Domain from Formula	Graph How Make?	Statement How Make?	Total Score
Possible Points	2	2	2	2	6	2	2	4	2	2	2	13
Mean	1.20	1.60	1.40	1.20	4.20	0.80	0.00	0.80	1.00	0.80	1.80	9.80
Standard Deviation	0.45	0.89	0.55	0.45	1.43	1.10	0.00	1.10	0.00	0.45	0.45	0.84

Statistics for Differences	Describe	Statement Why	Statement Domain	Statement Range	Statement Total	Formula Y/N	Formula Why	Formula Total	Domain from Formula	Graph How Make?	Statement How Make?	Total Score
Mean	0.20	1.40	0.60	0.80	2.80	-1.20	-0.80	-2.00	-0.20	0.20	0.80	1.80
Standard Deviation	0.45	0.89	0.89	0.84	1.64	1.10	0.45	1.41	0.84	0.84	0.84	2.17
Count	5	5	5	5	5	5	5	5	5	5	5	5
t Ratio	1.000	3.500	1.500	2.138	3.810	-2.449	-4.000	-3.162	-0.535	0.535	2.138	1.857
Degrees of Freedom	4	4	4	4	4	4	4	4	4	4	4	4
p-value	0.1870	0.0124	0.1040	0.0497	0.0095	0.9648	0.9919	0.9829	0.6893	0.3107	0.0497	0.0685
	Accept H_0			Reject H_0 in favor of H_A				Accept H_0	Accept H_0	Accept H_0	Reject H_0	Reject H_0 in favor of H_A

Hypothesis Testing

H_0: The mean difference between teaching students functions the 'traditional' way and 'technology' way is zero

H_A: The mean difference between teaching students functions the 'technology' way is greater than the 'traditional' way

Reject H_0 in favor of H_A if p-value < 0.10

Data shows that these students had some knowledge of the definition of a function on both the Midterm and the second Quiz. Darren was the only student to improve his test scores from a "1" to a "2" by properly defining a function on the Quiz #2 (Blue). The largest improvement came on functions given as statements where all five students showed improvement by increasing their average from 1.4 points to 4.2 points out of 6. The group really struggled when asked whether the step function from the second Quiz gave a well-defined function for all real numbers. Three of the five students thought that this could not be a function because $h(x)$ equaled two different formulas and not one. The other two students claimed that this was a function because there were no division symbols in either part of the formula and thus all real numbers can be inserted. Both tests asked to determine the domain of a function given by a formula and both tests had a square root symbol in it. The formula on Quiz #2 had the square root symbol in the denominator. Two of the students stated that the denominator could not be zero and the other three stated that the denominator could not be negative. No student combined these two observations to properly identify the complete domain of this formula.

The Midterm and Quiz #2 (Blue) both asked how to modify a graph and a statement to make it represent a function. No student provided a valid example on either test and most students received a "1" for claiming that the given graphs could not be functions because they failed the VLT. Darren was the only student to not correctly state on Quiz #2 that in order for a given statement to be a function you had to limit the number of roles an actor can play to one. He identified the domain and range correctly but he reversed the correspondence and said that this would be a function if you "assign(ed) every role to 1 actor only."

The total score for each of these students increased between the Midterm and Quiz #2. We used 90% confidence level to see whether this increase was significant.

H_0 = the mean difference between teaching students functions the 'traditional' way and 'technology' way is zero.

H_A = the mean difference between teaching students functions the 'technology' way is greater than the 'traditional' way.

We compared both the total score received from each participant on the Midterm to the score received on the Blue Quiz #2, as well as the total scores received on each type of question individually. The results of the Hypothesis Testing are shown as part of Figure 20.

Comparing the Total Score from the Midterm to the Blue Quiz #2, with 90% confidence we can say that students responded, learned, and performed better on tests when taught functions using 'technology' after being taught using a 'traditional' method. Just like the other group who took the Yellow Quiz #2, their scores improved statistically. Their average improvement was only 1.8 points which is still small when compared to the perfect score of 18 points. The average score for Quiz #2 (Blue) was only 9.8 points out of 18 possible points, a 54% average. The 90% confidence interval for the true mean difference based on the formula in Figure 14 is $1.80 \pm 1.486 = (0.31, 3.29)$. Even though the average student improved as a result of the 'technology' lecture, something is still missing from the way functions are taught that is necessary to build a complete concept image of functions. Interestingly, when comparing the test results on the different representations of functions, this group significantly improved only on the statement representations of functions, while the students who took the Yellow Quiz #2 did not improve significantly in this area.

CHAPTER 6

Follow Up Interviews:

After grading the Final, I interviewed two students who were able to communicate their thoughts clearly and also struggled on the Midterm. These students also took the Yellow Quiz #2 as well as struggled on the Final. This interview required them to explain what they were thinking and what elements of the definition of the function were not connecting for them. It was also critical for these students to verbally communicate what they didn't understand. Each interview was conducted separately so that they would not influence each other's answers. The importance of understanding where these students struggled is that it may give educators some insight into tailoring future lectures on functions.

The interview was broken down into three main sections. First I asked the students a difficult function question about whether or not a formula represented a function for all real numbers. I asked them to explain their answer as well as identify the domain. The formula in this question was a composition of two different formulas, $(f \circ g)(x)$. One of the formulas required division and the other included an expression under a square root. Since these two students did not demonstrate their understanding of this concept on similar questions earlier, I did not expect them to answer this question quickly or correctly. The second part of the interview included a preselected set of questions asked when the student struggled. These questions were designed to force the student to break the main problem down into questions about the functions $f(x)$ and $g(x)$ and then apply that knowledge to the composition. Each student was given paper to write on to answer any of the supporting questions I asked them. The third and final part of the interview occurred after the students answered the first question correctly and claimed to understand the function concept. At this point I asked the students to generate two different functions, make a composite function and find the domain.

Throughout the interview, I asked each student to explain where they were struggling. I used phrases like "please explain to me what you are doing now" to help the students communicate their thoughts. My goal was to wait for the student to answer or ask for help before proceeding. For instance, if the student had trouble understanding what $(f \circ g)(x)$ meant, then I first asked them questions about $f(x)$ and $g(x)$ and then gave them a handout of a function machine representation of $f(x)$ and $g(x)$. Once they understood the individual functions, I asked them what $(f \circ g)(x)$ represents. If they were unsure of the notation, $(f \circ g)(x)$, I showed them a function machine representation of the process described by $(f \circ g)(x)$. When the student claimed to understand the notation used in the question, I asked them to explain what "well-defined" means. Then I asked them to determine whether $f(x)$ and $g(x)$ represent well-defined functions for all real numbers. In addition to the above questions that I expected the students to struggle with, I also had pre-written handouts on understanding all real numbers, the domain of a function, and the definition of a function. The interview template including the questions and answers that I used during the Interview is shown in Appendix H.

During the interviews, both students did not immediately respond to the main question and did not recognize two functions that need to be understood independently to start working on the problem. I walked both students through the pre-determined scenario and they struggled where I thought they would struggle; the notation and identifying the "bad points" in the domain. Both students didn't respond to any of my questions until I asked the shorter straightforward questions requiring "yes" or "no" answers. Then after getting them involved, I was able to walk them back through each step in the process to solve the original problem.

Each student lacked confidence in dealing with formal mathematical notation. They seemed to get lost in the symbols used and appeared to be distracted by the notation which prevented them from understanding the question being asked. Both students seemed to think math was simply applying rules to a set of numbers. In both interviews it was clear that the students knew the elements of the definition of a function but they did not know how to put them together. Each student was able to understand questions when broken down into small and specific pieces, but when asked how these pieces related to the main question, they struggled. Definitely, the missing piece of their cognitive roots in this context was the ability to understand the notation and the ability to decompose a complex problem into simpler ones. This suggests that instructions for these types of students should concentrate more on various ways to describe and denote functions, as well as on algorithms for simplifying harder problems.

CHAPTER 7

Conclusions:

As discussed earlier, functions are one of the first abstract concepts students are faced with in mathematics and this topic typically begins their transition to higher mathematical thought. In higher level courses, students are no longer able to simply mirror basic examples by changing the numbers. They have to think about the definitions of theorems and be prepared to explain their answers. In the case of functions, they have to understand the definition of a function and then apply that to different representations. We as educators can smooth this transition to higher level mathematics by teaching students in various effective ways. In this paper we studied the effects of a PowerPoint lecture that contained animations and an applet [8] movie on functions. The research discussed in Chapter 2 shows that visualizations produce a better concept image for the students and will ultimately help smooth the transition to higher level mathematics. This implies that all educators need to incorporate these contemporary techniques into their instructions to make sure that no students get left behind when abstract concepts are introduced.

Our case study compared ‘traditional’ and ‘technology’ focused teaching methods and asked the students similar questions about functions throughout the semester to gauge their understanding of the topic. Each test asked the students to give a definition of a function and then determine whether different scenarios represented functions as well as identify the domain and range for these scenarios.

The findings from this case study “provide a picture of the existing situation, not a picture of what could be achieved under dramatically changed instruction ([2] page 44).” Perhaps future students would be able to increase their understanding of the function concept if there were better visualizations. With the advancement of technology and the eventual transition to computer based learning, better visualizations are bound to arise that are able to develop the concept image of each student more effectively. Also, if there were more hands on explorations of functions introducing the topic then more students may develop deeper cognitive roots. “Almost everyone involved in the teaching and learning of mathematics holds that the learning of mathematics is a personal matter in which learners develop their own personalized notions of mathematics as a result of the activities in which they participate ([2] page 44).”

Other ways to help smooth the transition to higher level mathematics may be to change the way we introduce the function concept to our students. The earlier kids can be exposed to components of functions, the more experience they will have when the mathematical definition is introduced in algebra. The participants in this case study had a very hard time interpreting the statement representations. All students need to be exposed to word problems as early as possible. The other main issue reflected in the students’ responses for the domain and range questions was their inability to consider non-numerical correspondences. Often students would claim the domain and range equaled all real numbers for a problem with non-numerical values. It is crucial to do non-numerical examples while introducing concepts because later learners are faced with solving various problems and they need the experience of feeling comfortable with different scenarios. Additionally, the word “function” is used in our everyday language and we need to use this experience in language to form a bridge between the “mathematical meaning and application of those same ideas ([6] page 139).” Perhaps there are even new ways to approach the function concept altogether that no one has thought of yet.

Our case study showed small improvements in students' learning but their interest in mathematics increased significantly only after the 'technology' lecture. They were more engaged during the 'technology' lecture and often asked questions and took notes. This was a major difference since most of the time during 'traditional' lectures the students were quiet and didn't write anything down. Also since the students were more engaged, I was more positive when presenting the information because "the reaction of students is a strong factor influencing a teacher's portrayal of the nature of mathematics in class ([2] page 43)." Additionally, the classroom discussion and flow of material was much easier to plan because the visual displays were already prepared and I could interact more with the students and gauge their reactions. The learners paid more attention because I was facing them the whole time and never had my back turned to them. I recommend using PowerPoint lectures and animations for mathematics courses, especially algebra. I also recommend using visual materials as a complement to the textbook, not as a substitute for it.

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APPENDIX A

Pre-Test Function Questions:

On the following page is the Pre-Test that I gave to all the students in my section the first week of school, prior to covering the material on Functions. The results of this test were used as the baseline to compare future test scores against. (Pre-Test)

Function Quiz

1) Background information:
   a) What is your Name? ______________________________________
   b) What is your Major? ______________________________________
   c) What is the highest level of Math class you have ever taken? _________________________
   d) How long ago did you take this class? __________________________
   e) What grade did you receive in this class? ________________

2) Explain what a function is.

3) Determine if these are functions (please circle Yes or No and explain why or why not for each)
   a) An insurance company assigns to every costumer their phone number as an id number. Is this procedure a function?

   Yes / No
   Why or Why Not?

   Domain/Range

   b) On official transcripts, California State University Channel Islands gives every graduating student his/her university GPA. Is this a function?

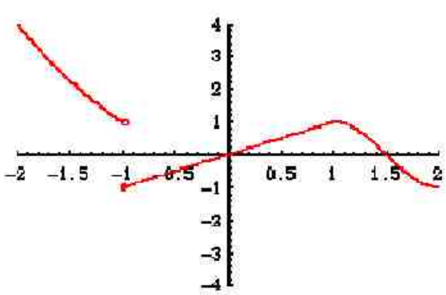
   Yes / No
   Why or Why Not?

   Domain/Range

   c) Does the formula $f(x) = \frac{3}{(x-3)^2}$ give a well-defined function for all real numbers?

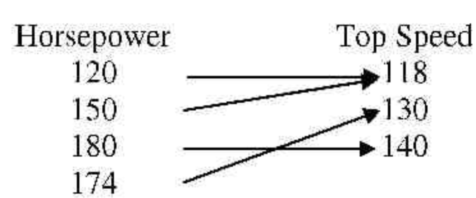
   Yes / No
   Why or Why Not?

   d) Does this graph represent a function?



   Yes / No
   Why or Why Not?

   e) Does this table represent a function?



   Yes / No
   Why or Why Not?

APPENDIX B

Quiz #1 Function Questions:

On the following pages is the first quiz that I gave the students in the 3rd week of class, following the ‘traditional’ lectures on the white board. Only my Math 95 class took this test. Only questions 1, 2, 3c, 3d, 5b, and 6 are applicable to this case study. The other questions asked on this test cover the other material found in Chapter 7 of the text book. (Quiz #1)

Chapter 7 Quiz

Name:___________________________________

1. Explain what a *function* is.

2. Determine if these are *functions* (please circle Yes or No and explain why or why not for each)

   a) The correspondence that assigns to each person of a Rock Band the instrument that they can play?

      Yes / No
      Why or Why Not?

      What is the Domain?

      What is the Range?

   b) The correspondence between a person and their Birthday

      Yes / No
      Why or Why Not?

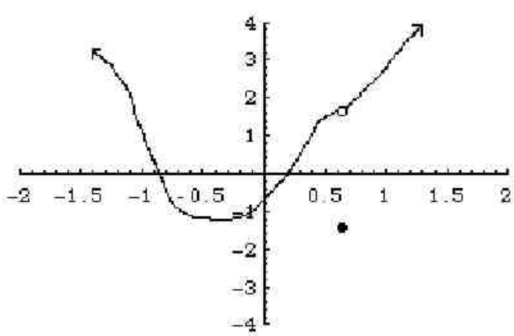
      What is the Domain?

      What is the Range?

   c) $x^2 + y^2 = 1$

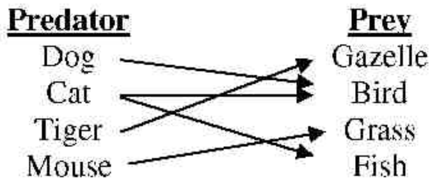
      Yes / No
      Why or Why Not?

   d)



      Yes / No
      Why or Why Not?

e)

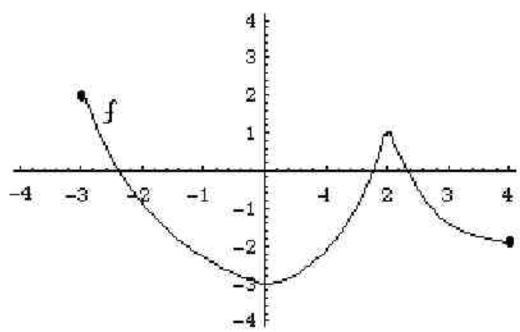


Yes / No
Why or Why Not?

f) $\{(-1,3), (0,2), (8,4), (0,5)\}$

Yes / No
Why or Why Not?

3) From this Graph,



a) Determine $f(2)$.

b) What *x-value* gives $f(x) = 2$?

c) What is the Domain of f?

d) What is the Range of f?

4) The amount A of coffee that suppliers are willing to supply at price p is given by $A(p) = -2.5p + 26.5$. What is the domain of the function?

5) If $g(x) = \dfrac{2x-1}{x+5}$,

a) Find $g(t\text{-}2)$.

$g(t\text{-}2) =$

b) What is the Domain of g?

6) Determine the Domain and Range of this function,
{(0,1), (3,7), (-1, 3), ($,@)}

a) Domain =

b) Range =

7) Find $f(x)$ and $g(x)$ such that $h(x) = (g \circ f)(x)$ when $h(x) = (3x-1)^2$

$f(x) =$

$g(x) =$

8) Find an equation of variation in which y varies jointly as x and z and inversely as w, and $y = 3$ when $x = 2$, $z = 3$, and $w = 4$.

9) If $f(x)=\dfrac{1}{(x-1)^2}$ and $g(x)=2x+3$ find,

a) $(g+f)(0)=$ $\boxed{}$

b) $(f-g)(x)=$ $\boxed{}$

c) $(f\cdot g)(x)=$ $\boxed{}$

d) $(g/f)(-1)=$ $\boxed{}$

e) $(g\circ f)(2)=$ $\boxed{}$

f) $(f\circ g)(x)=$ $\boxed{}$

g) What is the domain of $(f+g)(x)$? $\boxed{}$

h) What is the domain of $(f/g)(x)$? $\boxed{}$

10) In 1990, the life expectancy of females was 78.8 years. In 2000, it was 79.5 years. Let $E(t)$ represent life expectancy and t the number of years since 1990.

a) Find a linear function that fits the data.

b) Use the function from part (a) to predict the life expectancy of females in 2010.

c) When will females expect to live for 90 years?

APPENDIX C

Midterm Function Questions:

This midterm contained information from the fist third of the class and was administered during the 8^{th} week of classes. It was given and assessed to determine any long term memory of the function concept. All the questions listed here were on the Midterm as well as other questions from Chapter 8 and 9 in the text book. Questions 5a through 5e are not applicable to this case study. (Midterm #1)

Midterm Function Questions

1) Explain what a *function* is.

2) Determine if these are functions (please circle Yes or No and explain why or why not for each)
   a) The correspondence that assigns a price to a bar code on a product in a store?

      Yes / No
      Why or Why Not?

      What is the Domain?

      What is the Range?

   b) A bank uses your dog's name as the answer to a security question in case you forget your password. And if you have more than one dog, they combine their names to form one larger name. Does this process describe a function?

      Yes / No
      Why or Why Not?

      What is the Domain?

      What is the Range?

   c) Do these points describe a function?
      {(-5,4), (0,2), (7,7), (@,#)}

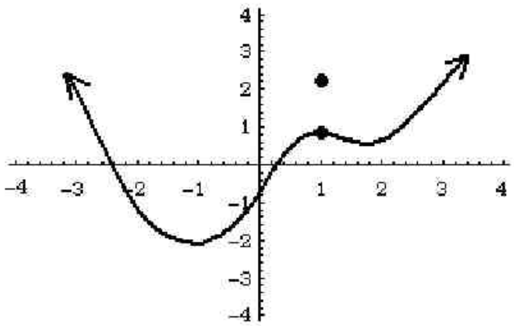
      Yes / No
      Why or Why Not?

      What is the Range?

   d) Does this formula represent a function?
      $y = \pm 5x$

      Yes / No
      Why or Why Not?

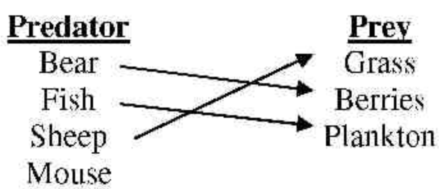
e) Does this graph represent a function?



Yes / No
Why or Why Not?

What is the Range?

f) Does this table describe a function



Yes / No
Why or Why Not?

What is the Domain?

3) What is the Domain of $s(t) = \dfrac{2t+1}{3t-9}$?

4) What is the Domain of $f(x) = \sqrt{x-6}$?

7) Determine the Domain and Range of this function,
{(a,j), (2,7), (-6,7), (*,!)}

Domain =

Range =

8) If $f(x) = 2x + 1$ and $g(x) = x + 3$ find,

a) $(f - g)(x) =$

b) $(f - g)(1) =$

c) $(g \cdot f)(-2) =$

d) $(f \circ g)(0) =$

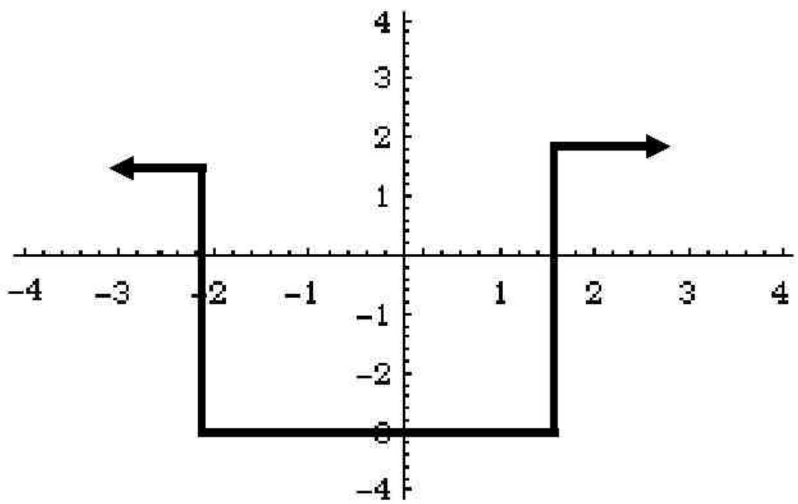
e) $(g \circ f)(x) =$

f) Does the formula $\left(\frac{f}{g}\right)(x)$ give a well-defined function for all real numbers?

Yes / No
Why or Why Not?

Extra Credit

I. Can you think of any way to make this graph represent a function? Explain.



II. As we have seen from the Chapter 7 quiz, the correspondence that assigns to each person of a Rock Band the instrument that they can play, does not represent a function. What ways can you think of to make this correspondence a function?

APPENDIX D

Quiz #2 Function Questions (Yellow):

This quiz was given after the PowerPoint lecture on functions to see if the students were able to understand the function concept better than when the material was presented to them the 'traditional' way. There were two different quizzes given here. The Yellow test was given to those students that did not perform well on the first quiz and/or the midterm. The questions on it were very similar to those that the students had seen before and had not been able to answer correctly. Also, I asked some questions to determine the students' preferences on lecture style and also to gather some additional information. (Quiz #2 Yellow)

Function Quiz 2

What is your Name? ______________________________________
Did you take notes during the last lecture? ____________________
Did you take notes in the previous lectures? __________________
Did you feel more comfortable last lecture or the previous lectures? ____________
Which types of notes do you like better, your own or the prepared ones? ______________________
Do you feel that you know more now after having the PowerPoint lesson? ____________
How much time do you spend on the computer on an average day? ____________________
What version of the Book are you using, New or Old? _________________

1) Explain what a *function* is.

2) Determine if these are *functions* (please circle Yes or No and explain why or why not for each)
   a) When a movie director casts a role for a part in their movie, they assign an actor to each character in the screenplay. Does this describe a function?

      Yes / No
      Why or Why Not?

      What is the Domain?

      What is the Range?

   b) If you make $10 an hour and you are not allowed to work overtime, does the correspondence between the number of hours you work and the amount of money you make describe a function?

      Yes / No
      Why or Why Not?

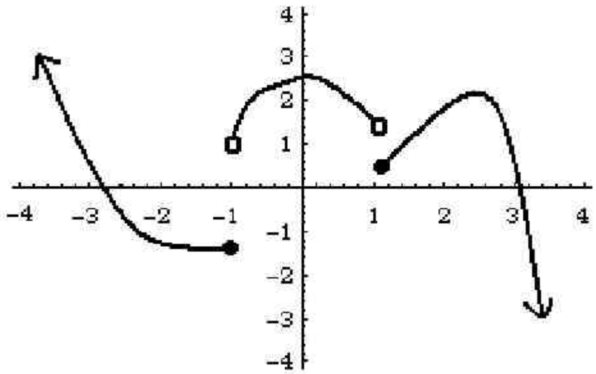
      What is the Domain?

      What is the Range?

   c) Does the formula $g(x) = \frac{2x+4}{(x-5)^2}$ give a well-defined function for all real numbers?

      Yes / No
      Why or Why Not?

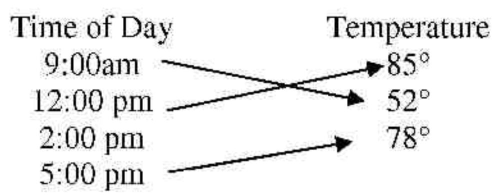
d) Does this graph represent a function?



Yes / No
Why or Why Not?

What is the Range?

e) Does this table represent a function?



Yes / No
Why or Why Not?

What is the Domain?

What is the Range?

APPENDIX E

Quiz #2 Function Questions (Blue):

This quiz was given after the PowerPoint lecture on functions to see if the students were able to understand the function concept better than when the material was presented to them the 'traditional' way. There were two different quizzes given here. The Blue test was given to those students that did perform well on the first quiz and/or the midterm. The questions on it were used to determine if these students could apply their knowledge of functions to more challenging questions then the ones they already showed competence in. Also, I asked some questions to determine the students' preferences on lecture style and also to gather some additional information. (Quiz #2 Blue)

Function Quiz 2

What is your Name? ______________________________________
Did you take notes during the last lecture? ___________________
Did you take notes in the previous lectures? ___________________
Did you feel more comfortable last lecture or the previous lectures? ____________
Which types of notes do you like better, your own or the prepared ones? ______________________
Do you feel that you know more now after having the PowerPoint lesson? ____________
How much time do you spend on the computer on an average day? _____________________
What version of the Book are you using, New or Old? _________________

1) Explain what a *function* is.

2) Determine if these are *functions* (please circle Yes or No and explain why or why not for each)
   a) Does the formula h(x) below, give a well-defined function for all real numbers?

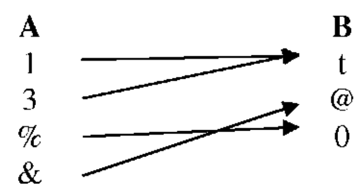
$$h(x) = \begin{cases} 2x+5, & \text{for } x \le -1, \\ 3x^2, & \text{for } x \ge -1, \end{cases}$$

      Yes / No
      Why or Why Not?

   b) Do both of these represent the same function?

      {(1,t), (%,0), (&,@), (t,3)}

      Yes / No
      Why or Why Not?



   c) What is the domain of this function, $h(t) = \dfrac{2t-1}{\sqrt{3t-6}}$?

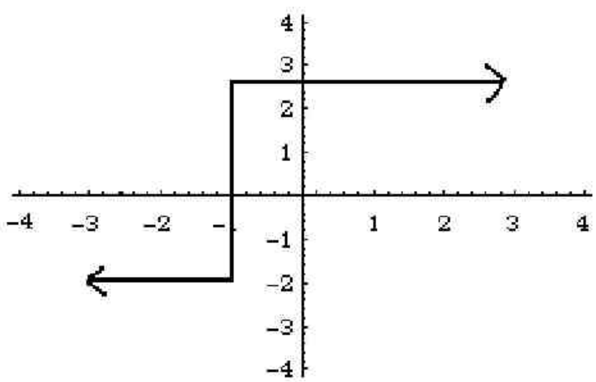
d) When a movie director casts a role for a part in their movie, they assign an actor to each character in the screenplay. Why is this not a function?

What is the Domain?

What is the Range?

How could you make it a function?

e) Could this graph represent a function? Explain.



APPENDIX F

Final Function Questions:

These questions were given as part of the Final exam and were intended to see if the students could now show a long term understanding of the function concept. The Final Exam was a common multiple choice exam given to all sections of the Intermediate Algebra course and in order to get written responses, these questions were given as Extra Credit. (Final)

Final - Extra Credit

Name: ___________________________________

Instructor: ________________________________

1) Explain what a *function* is.

2) An insurance company assigns to every customer an id number that matches the person's cell phone number. Does this procedure represent a function?

   Yes / No
   Why or Why Not?

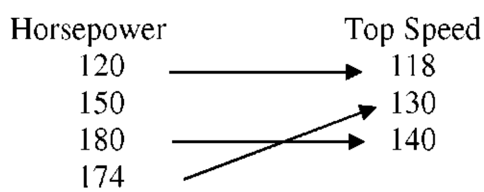
   What is the Domain?

   What is the Range?

3) Does the formula $h(x) = x^4 + x - 16$ give a well-defined function for all real numbers?

   Yes / No
   Why or Why Not?

4) Does this table represent a function?



   Yes / No
   Why or Why Not?

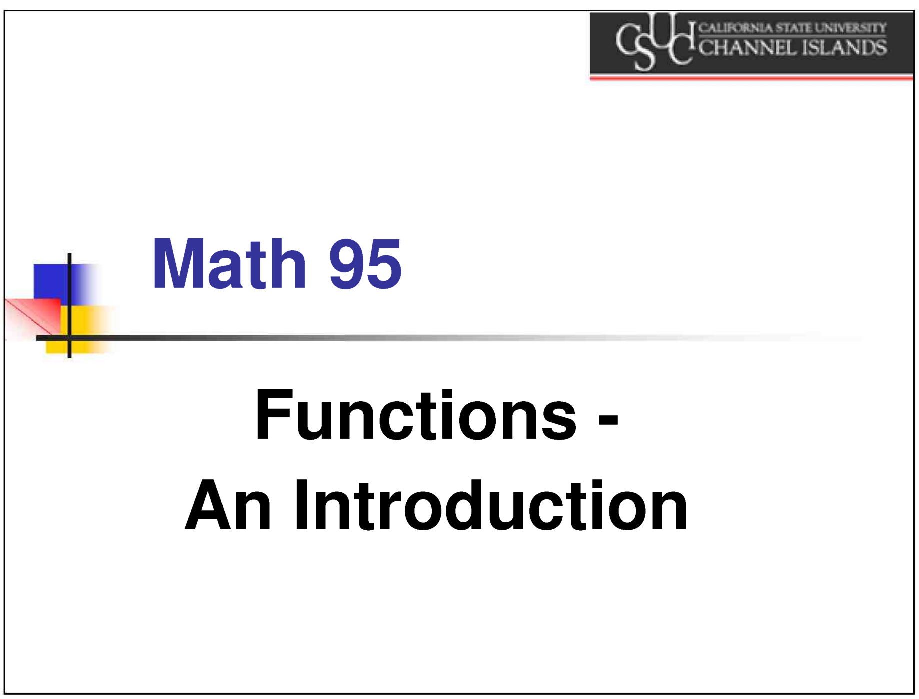
   What is the Domain?

   What is the Range?

APPENDIX G

PowerPoint Lecture:

For the technology portion of my case study, I used PowerPoint to display and animate the concepts of a function. I also put a hyperlink to an applet [8] movie that explains how a function machine works. I used a laser pointer and a remote mouse to control the slides so I could focus on the students and their reactions. Each line or bullet filled in as I talked as if I was writing it on the board myself. I had complete control of what the students saw and the pace at which they saw it. I removed some of the key terms from the slides and then handed them out to the students so they could follow along and fill them in. This gave the students notes that they could take home and study. Also at the end of the student copy, I gave them the homework questions so they could practice what they just learned. I have included the screenshots from the lecture and the matching slides from the student copy below. (PowerPoint Lecture, PowerPoint Student Copy)



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Real World Relationships

- In the physical world there are often relationships between one property and another.
  - For instance,
    - the outside temperature corresponds to the time of day,
      - 12 Noon is hotter than, 12 midnight.
    - the size of your car's engine is linked to it's fuel economy,
      - The bigger your car's engine, the less fuel economy it gets.

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Relation (Definition)

- A *relation* is a correspondence between a first set, called the *domain*, and a second set, called the *range*, such that each member of the domain corresponds to *at least one* member of the range.

Student Copy

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Real World Relationships

- In the physical world there are often ______ between one property and another.
  - For instance,
    - the outside temperature corresponds to the ______
      - 12 Noon is hotter than, ______
    - the ______ of your car's engine is linked to it's fuel economy,
      - The ______ your car's engine, the less fuel economy it gets.

2

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Relation (Definition)

- A ______ is a correspondence between a first set, called the ______, and a second set, called the ______, such that each member of the domain corresponds to ______ member of the range.

3

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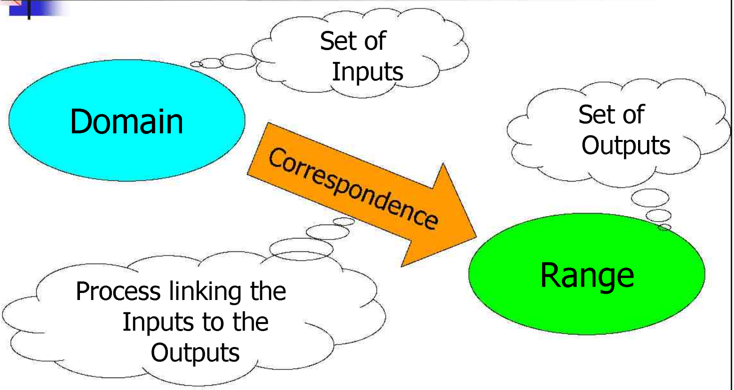
Relation (3 Parts)



4

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Function – a Special Relation

- If the relation between the two sets is such that each element in the domain corresponds to *only one* element in the range, then we call this relation a *function*.
  - i.e., a *function* is a correspondence between two sets where each input has *only one* output.

5

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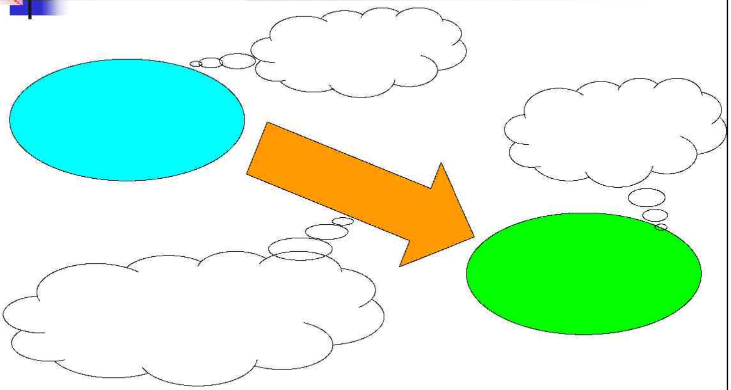
Relation (3 Parts)



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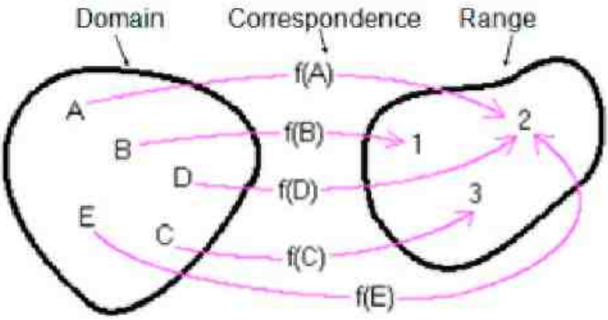
Function – a Special Relation

- If the relation between the two sets is such that each element in the ________ corresponds to ________ element in the ________, then we call this relation a ________
  - i.e., a *function* is a correspondence between ________ where each input has ________ output.

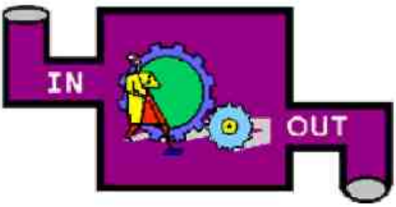
5

Function Overview



Function Visualization

- There is another useful visualization for thinking about functions, and that is called the…
- Function Machine



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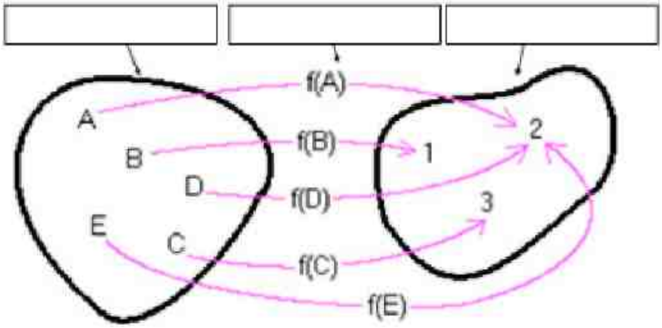
Function Overview



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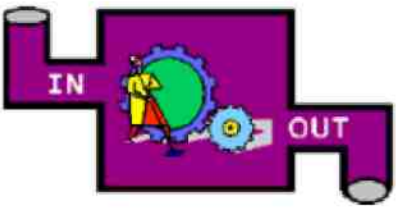
Function Visualization

- There is another useful visualization for thinking about functions, and that is called the…
-



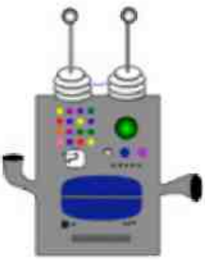
7

Function Machine

- The *domain* of the function is the set of all possible inputs to the Machine.
- The Machine itself carries out the process of converting elements of the domain into elements of the range.
- The *range* is the set of all possible outputs of the Machine.

8

Function Machine Movie



http://207.207.4.198/pub/flash/9/9.html

9

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Function Machine

- The ______ of the function is the set of all possible inputs to the Machine.
- The Machine itself carries out the process of ______ elements of the domain ______ elements of the range.
- The ______ is the set of all possible outputs of the Machine.

8

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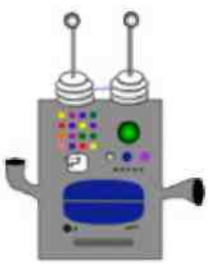
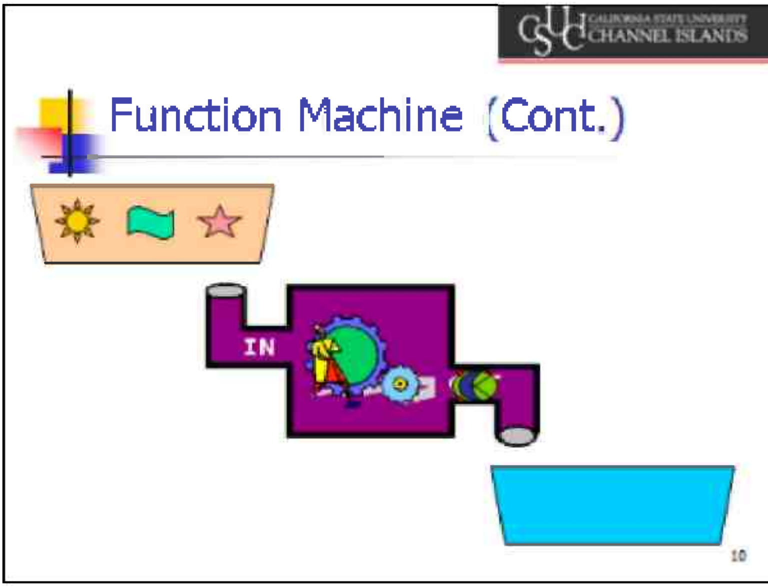
Function Machine Movie



http://207.207.4.198/pub/flash/9/9.html

9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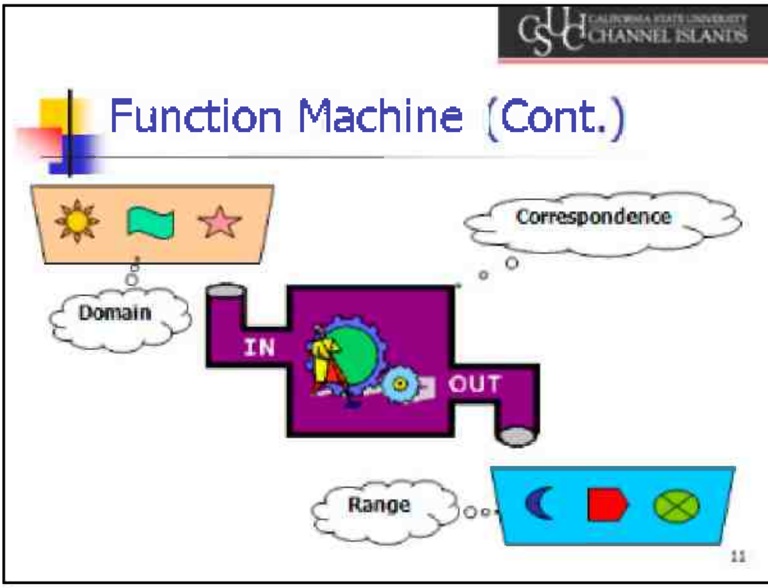
Function Machine (Cont.)
IN
10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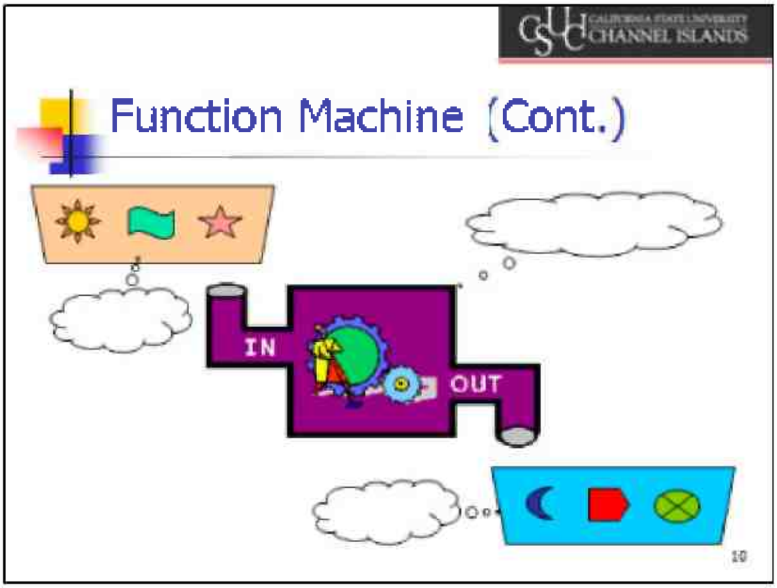
Function Machine (Cont.)
Correspondence
Domain
IN
OUT
Range
11

Student Copy



Function Machine (Cont.)

- Since both the *domain* and the *range* are sets, you can use set-builder notation to list their elements.
- Therefore, from the previous slide, the
  - Domain is 
  - And the Range is 

12

Other Representations

- That same data from the previous slides can be represented in other, equivalent, ways.
  - As a series of points

- Where the elements of the domain are listed first and are paired with their corresponding values of the range, which are listed second.

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Function Machine (Cont.)

- Since both the *domain* and the *range* are ____, you can use set-builder notation to list their ____
- Therefore, from the previous slide, the
  - ____ is {  }
  - And the ____ is {  }

11

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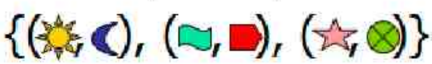
Other Representations

- That same data from the previous slides can be represented in other, ____ ways.
  - As a series of ____



- Where the elements of the domain are listed ____ and are paired with their corresponding values of the ____ which are listed second.

12

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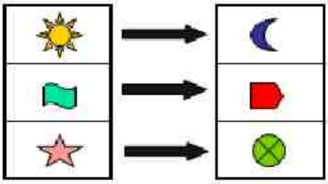
Other Representations (Cont.)

- Or as a table



- Where the elements of the domain are listed in the first column and are paired with, using arrows, their corresponding values of the range, which are listed in the second column.

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Question 1

- Do these points represent a *function*?

{(#,@), (!,$), (*,%), (3,5), (s,j)} Yes

- What is the domain and range?

Domain ={#, !, *, 3, s}

Range = {@, $, %, 5, j}

- Remember, to be function each element of the domain must only be assigned to one element of the range.
- Remember that the domain and range are a set of values, so you need to use set-builder notation to define them.

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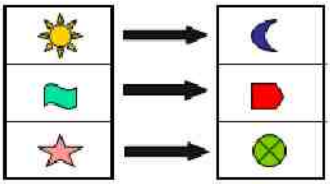
Other Representations (Cont.)

- Or as a ____



- Where the elements of the ____ are listed in the first column and are paired with, using arrows, their corresponding values of the range, which are listed in the ____ column.

13

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Question 1

- Do these points represent a *function*?

{(#,@), (!,$), (*,%), (3,5), (s,j)}

- What is the domain and range?

- Remember, to be function each element of the domain must only be assigned to ____ element of the range.
- Remember that the domain and range are a ____ of values, so you need to use set-builder notation to define them.

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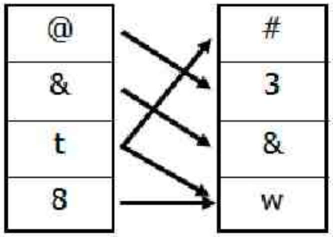
Question 2

- Does this table represent a *function*? No
- What is the domain and range?

Domain = {@, &, t, 8}

Range = {#, 3, &, w}



- Remember, to be function each element of the domain must only be assigned to one element of the range.
- Remember that the domain and range are a set of values, so you need to use set-builder notation to define them.

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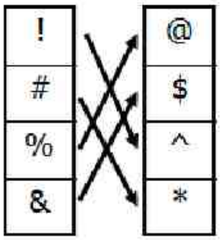
Question 3

- Do both of these represent the same function? Yes

{(#,*), (!,^), (%,@), (&,$),}



- What is the domain and range?

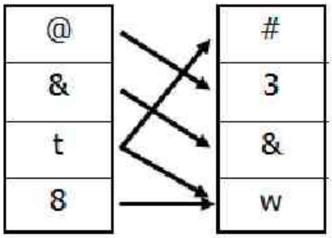
Domain = {#, !, %, &}

Range = {*, ^, @, $}

Student Copy

Question 2

- Does this table represent a *function*?
- What is the domain and range?



- Remember, to be function each element of the domain must only be assigned to ____ element of the range.
- Remember that the domain and range are a ____ of values, so you need to use set-builder notation to define them.

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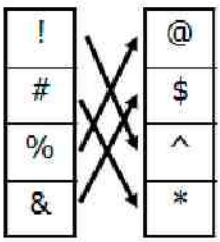
Question 3

- Do both of these represent the same function?

{(#,*), (!,^), (%,@), (&,$),}



- What is the domain and range?

Function Notation

- Function notation allows us describe the correspondence without having to draw a function machine, a list of points, or a table.
- We say that;

- Where the elements of the range are determined by the elements of the domain through a process of transformation. This is especially useful when dealing with numbers.

18

Function Machine for Equations

19

Function Notation

- allows us describe the correspondence without having to draw a function machine, a list of points, or a table.
- We say that;

- Where the elements of the are determined by the elements of the domain thro ocess of transformation. This is especially useful when dealing with numbers.

17

Function Machine for Equations

18

Presentation

Graphs

- Now that we have described a function using an equation with arbitrary inputs, we need to identify all the possible inputs and outputs.
  - *domain* and *range*
- This will generate a series of points.
- Then once we have these points, we can graph them on a standard x-y axis.

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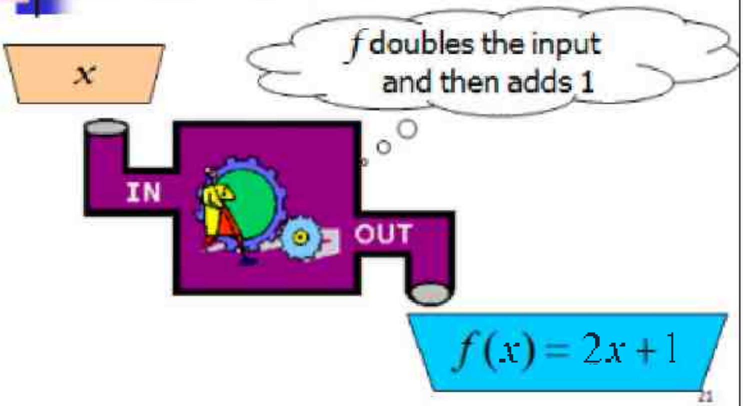
Example 1



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Graphs

- Now that we have described a function using an ________ with arbitrary inputs, we need to identify ________ possible inputs and outputs.
  - ________
- This will generate a series of ________
- Then once we have these points, we can ________ them on a standard x-y axis.

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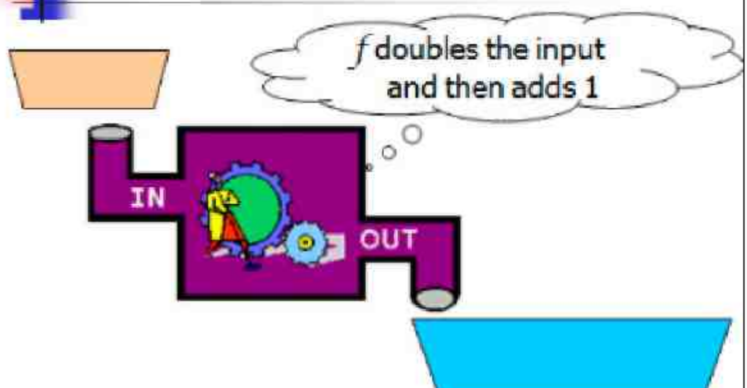
Example 1



20

Example 1 (Points)

- Now we can list some of the points in a table, using the process f which doubles the input and then adds 1 to get outputs of $f(x) = 2x + 1$.

  $f(-2) = 2(-2) + 1 = -3$

- However, you are not able to list all the points because there are an infinite number of them.

x	$f(x)$
-2	
-1	-1
0	1
1	3
2	5

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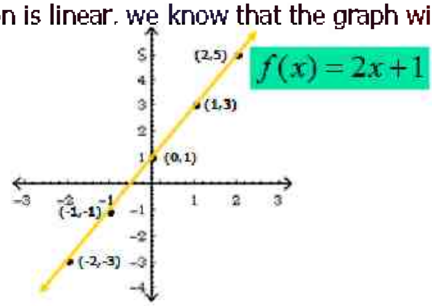
Example 1 (Graph)

- Then we can plot these points on a graph and connect the dots to represent all the possible elements of the *domain* and *range.*
- But how do we know how to "connect the dots"?
- Since the function is linear, we know that the graph will be a straight line.

x	$f(x)$
-2	-3
-1	-1
0	1
1	3
2	5



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Example 1 (Points)

- Now we can list ______ of the points in a table, using the process f which doubles the input and then adds 1 to get outputs of $f(x) =$ ______
- However, you are not able to list al the points because there are an ______ number of them.

x	$f(x)$

21

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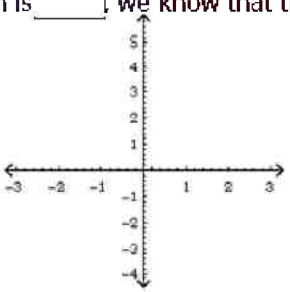
Example 1 (Graph)

- Then we can ______ these points on a graph and connect the dots to represent all the possible elements of the *domain* and *range*.
- But how do we know how to "______"?
- Since the function is ______, we know that the graph will be a straight line.

x	$f(x)$
-2	-3
-1	-1
0	1
1	3
2	5



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Determining the Domain From a Graph

- Since we know that a graph can represent a function,
  - and that the domain is all the input values of that function,
    - i.e. the x-values of the graph,
  - the domain is then all the possible x-values that have associated y-values.

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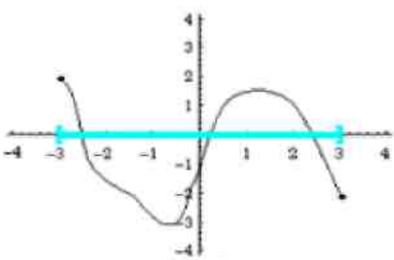
How to Find the Domain From a Graph

$$Domain = \{x \mid -3 \le x \le 3\}$$



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Determining the Domain From a Graph

- Since we know that a ______ can represent a function,
  - and that the domain is all the ______ values of that function,
    - i.e. the ______ of the graph,
  - the ______s then all the possible x-values that have associated y-values.

23

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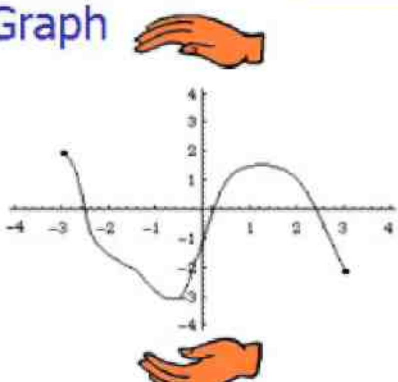
How to Find the Domain From a Graph



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Determining the Range From a Graph

- Similarly,
  - we know that the range is all the output values of the function,
    - i.e. the y-values of the graph,
  - the range is then all the possible y-values that have associated x-values.

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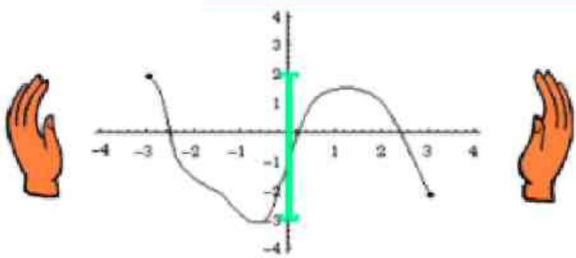
How to Find the Range From a Graph

$$Range = \{y \mid -3 \le y \le 2\}$$



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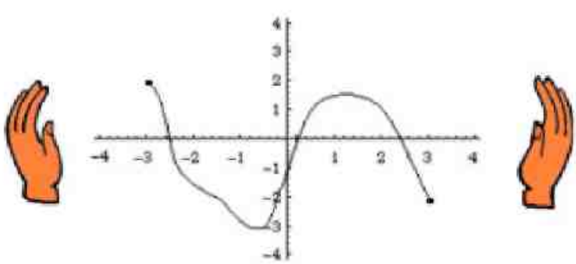
Student Copy

Determining the Range From a Graph

- Similarly,
  - we know that the range is all the _____ values of the function,
    - i.e. the _____ of the graph,
  - the _____ is then all the possible y-values that have associated x-values.

25

How to Find the Range From a Graph



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No Endpoints

- That works for graphs with endpoints, but what if the graph does not have endpoints?
- It will still work! You just have to remember that the arrows mean that the values go to infinity.

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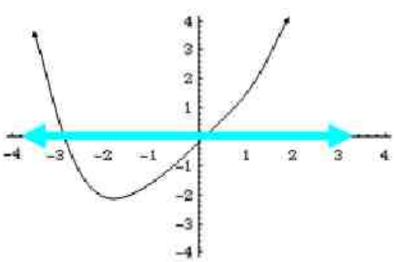
The Domain of a Graph Without Endpoints

$Domain = \{x | x \text{ is a real number}\}$



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No Endpoints

- That works for graphs with ________, but what if the graph does not have endpoints?

- It will still work! You just have to remember that the arrows mean that the values go to ________

27

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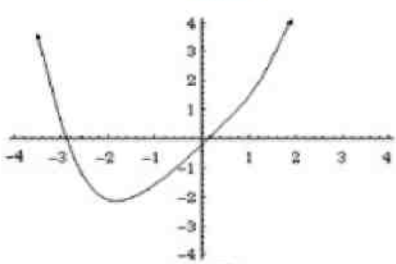
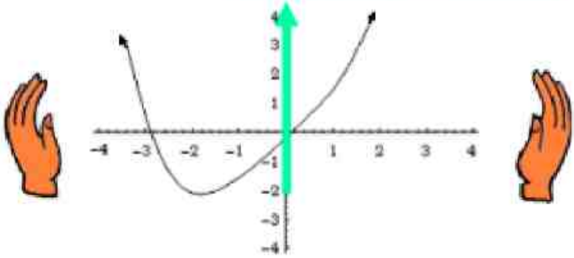
The Domain of a Graph Without Endpoints



28

The Range of a Graph Without Endpoints

$Range = \{y \mid y \geq -2\}$



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Graphs With Holes

- This technique works for graphs with and without endpoints, but what if the graph has holes, or missing points in it?
- It will still work! You just have to think of the graph without the holes and then remove them from the domain and range.

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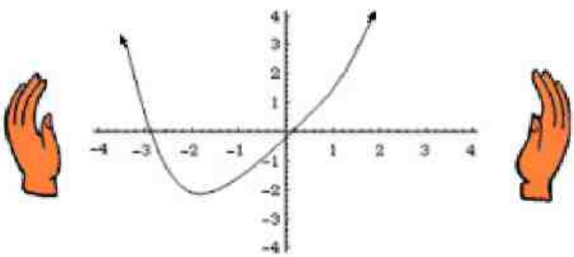
The Range of a Graph Without Endpoints



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Graphs With Holes

- This technique works for graphs with and without endpoints, but what if the graph has ________, or missing points in it?
- It will still work! You just have to think of the graph without the holes and then ________ them from the domain and range.

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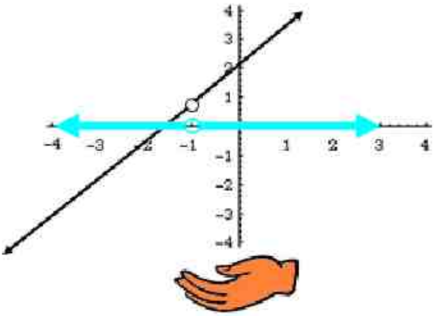
The Domain of a Graph With a Hole

$$Domain = \{x \mid x \text{ is real, but } x \neq -1\}$$



32

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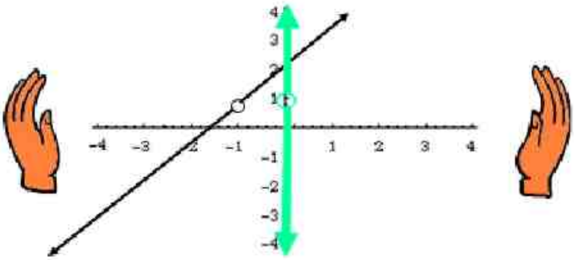
The Range of a Graph With a Hole

$$Range = \{y \mid y \text{ is real, but } y \neq 1\}$$



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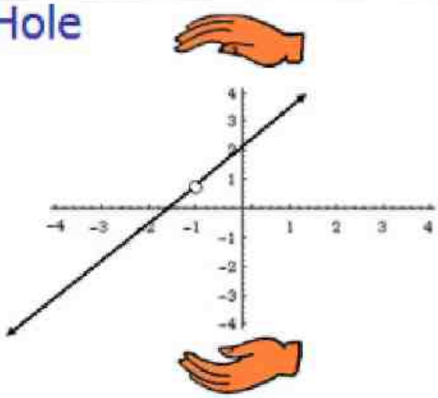
The Domain of a Graph With a Hole



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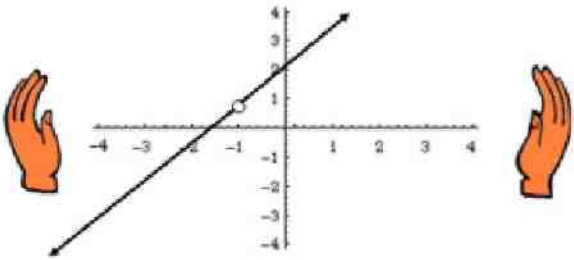
The Range of a Graph With a Hole



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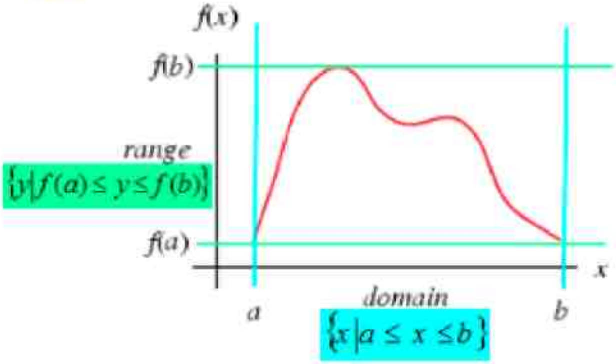
Another Way

- Instead of thinking about the 'shadow' the function leaves on the *x* or *y-axis*, you can also think about,

  A 'fence' that encompasses the graph around either the *x-axis*, for domain, or the *y-axis*, for the range.

Domain and Range Fence



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Another Way

- Instead of thinking about the ______ the function leaves on the *x* or *y-axis*, you can also think about,

  A ______ that encompasses the graph around either the *x-axis*, for ______, or the ______ for the range.

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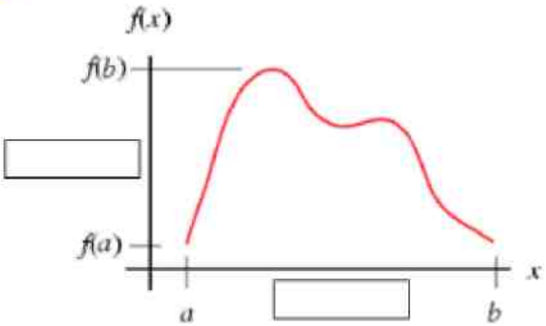
Domain and Range Fence



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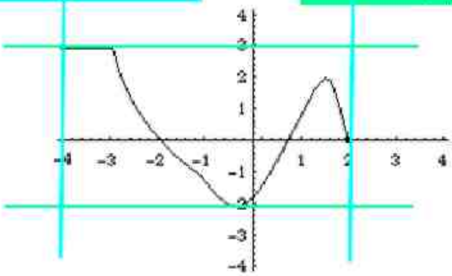
Question 4

- What is the domain and range of this function?

$Domain = \{x \mid -4 \le x \le 2\}$ $Range = \{y \mid -2 \le y \le 3\}$



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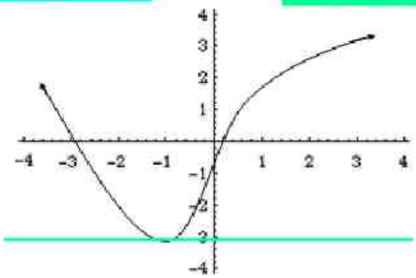
Question 5

- What is the domain and range of this function?

$Domain = \{x \mid x \text{ is real}\}$ $Range = \{y \mid y \ge -3\}$



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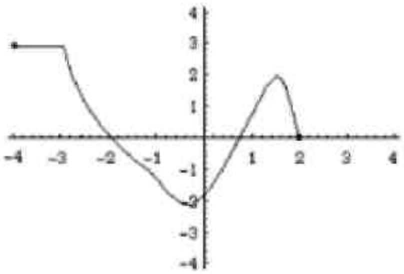
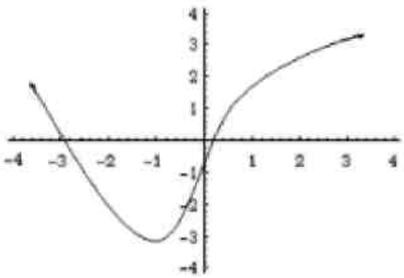
Question 4

- What is the domain and range of this function?



Question 5

- What is the domain and range of this function?



Points That Are Not a Function

- As we stated earlier, if a correspondence assigns an input value, say 1, to two different output values, say 2 and 4, then this relation cannot be a function.
- This means that graphically a function can never have a scenario where one input, or *x-value,* gives two outputs, or *y-values.*
  - i.e. you cannot have a graph with the points (1,2) and (1,4) on the same line.

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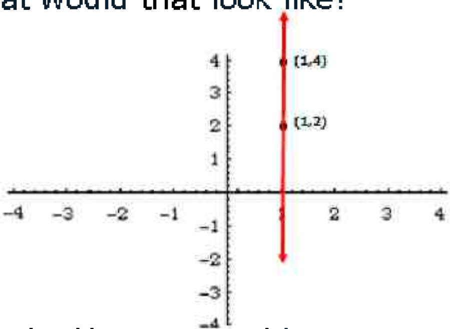
A Vertical Line

- What would that look like?



- It looks like a vertical line!

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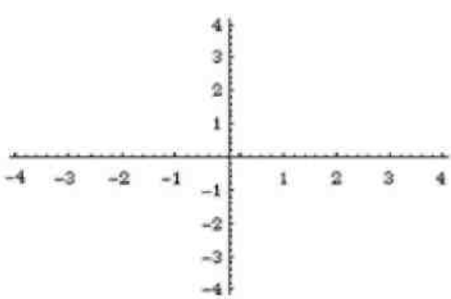
Points That Are Not a Function

- As we stated earlier, if a correspondence assigns an input value, say ____, to two different ________ values, say ____ and ____, then this relation cannot be a function.
- This means that graphically a function can ________ have a scenario where one input, or *x-value*, gives two outputs, or ________
  - i.e. you cannot have a graph with the points (1,2) and (1,4) on the ________ line.

37

A Vertical Line

- What would that look like?



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Vertical Line Test

- This means that a vertical line can never touch the graph of a function more than once, *otherwise*, there would be two outputs for one input.
- Now this is a very powerful tool that can be used to determine if a graph represents a function.
  - Because if a vertical line touches a graph only once, then that graph represents a function.

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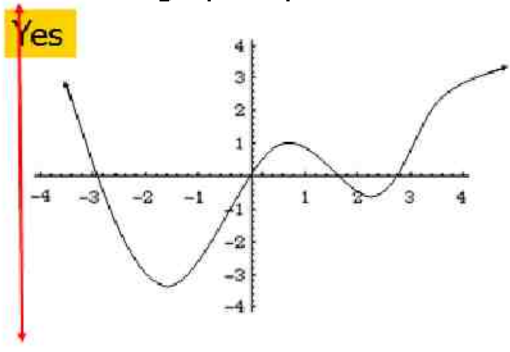
Question 6

- Does this graph represent a *function*?



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Vertical Line Test

- This means that a ________ can never touch the graph of a function more than ________, *otherwise*, there would be two outputs for one input.
- Now this is a very powerful tool that can be used to determine if a graph represents a ________
  - Because if a vertical line touches a graph only once, then that ________ represents a function.

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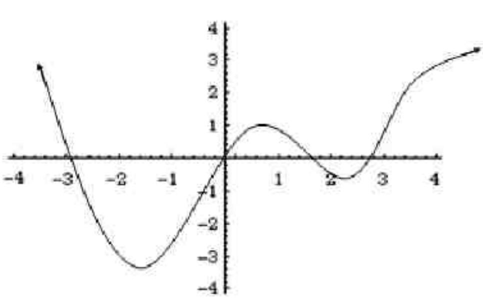
Question 6

- Does this graph represent a *function*?



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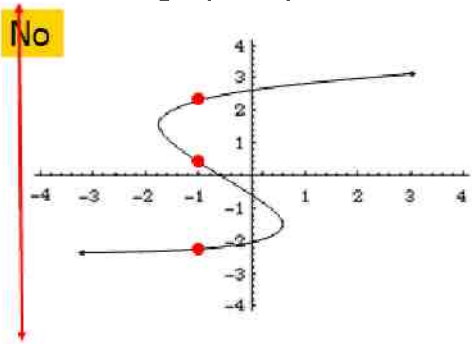
Question 7

- Does this graph represent a *function*?

No



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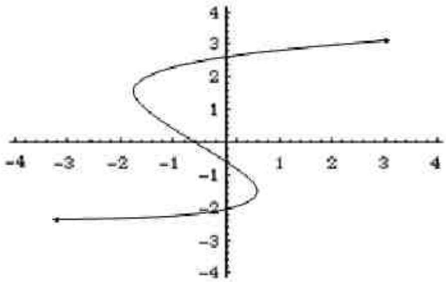
What is a Function?

- Recall that to be a *function*, every element of the input set, called the *domain*, must get assigned by the *correspondence* to only one element of the output set, called the *range*.

Question 7

- Does this graph represent a *function*?



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What is a Function?

- Recall that to be a *function*, every element of the input set, called the ________, must get assigned by the *correspondence* to ________ element of the output set, called the ________

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What is Not a Function?

- If any input can be assigned to two or more possible outputs, then the relation **is not** a function.
- If any element of the domain does not correspond to any element of the range, then that correspondence **does not** describe a relation, let alone a function.
  - That means that every input must go somewhere to be a relation.

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Question 8

- Does this statement represent a function?
  - The correspondence that assigns to each person in this class, their height in inches.

    Yes, because everyone has only one height.
- What is the domain?

  Set of students in this class
- What is the range?

  Set of positive measurements in inches

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What is Not a Function?

- If any input [ ] assigned to two or more possible outputs, then the relation [ ] a function.
- If any element of the domain [ ] correspond to any element of the range, then that correspondence [ ] [ ] describe a [ ] let alone a function.
  - That means that every [ ] must go somewhere to be a relation.

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Question 8

- Does this statement represent a function?
  - The correspondence that assigns to each person in this class, their height in inches.
- What is the domain?
- What is the range?

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Question 9

- Does this statement represent a function?
  - The correspondence that assigns to each author the title of the books that they have written.

    No, because some authors have written more than one book.
- What is the domain?

  Set of authors
- What is the range?

  Set of book titles

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Question 9 (Cont.)

- How could you make that statement a function?

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Question 9

- Does this statement represent a function?
  - The correspondence that assigns to each author the title of the books that they have written.

- What is the domain?

- What is the range?

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Question 9 (Cont.)

- How could you make that statement a function?

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How can you turn something that is not a function, into a function?

- Adjust the correspondence so that each element of the domain gets assigned to only one element of the range.
- Adjust the domain so that each input gets assigned to only one output.
  - This is done by removing the 'bad' points from the domain.
    - The 'bad' points are those that either don't have output values associated with them,
    - or they can get assigned to more than one output.

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Question 9 (Cont.)

- How could you make that statement a function?
  - You could change the correspondence to state that if the author wrote more than one book, then each author will only be assigned to the title of <u>their most recent</u> book.
  - Or you could limit your domain to be only those authors who have written <u>only one</u> book.

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How can you turn something that is not a function, into a function?

- Adjust the ____________ so that each element of the domain gets assigned to only one element of the ____________
- Adjust the ____________ so that each input gets assigned to ____________ output.
  - This is done by removing the ____________ from the domain.
    - The 'bad' points are those that ____________ don't have output values associated with them,
    - or they ____________ get assigned to more than one output.

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Question 9 (Cont.)

- How could you make that statement a function?

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Formulas

- Let's revisit *formulas*.
- They are the most common representation of functions that we will deal with in Algebra.
- Therefore, it is important to determine when a formula represents a function.

Example 2

- Suppose $f(x) = 3x^2 + 1$,
- Does $f(x)$ give a well-defined *function* for all real numbers?
  - i.e. Does $f(x)$ give real number outputs for all real number inputs?
- To answer this, we should look to the function machine for help.

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Formulas

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Example 2

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  - i.e. Does $f(x)$ give real number ________ for all real number ________
- To answer this, we should look to the ________ for help.

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Function Machine for Formulas

Example 2 (Cont.)

- Suppose you put any real number you want into this machine, say 2.
- Then what would you get out?

$$f(x) = 3(x)^2 + 1$$
$$f(2) = 3(2)^2 + 1 = 13$$

- Therefore $f(2) = 13$, which is a real number.

Function Machine for Formulas

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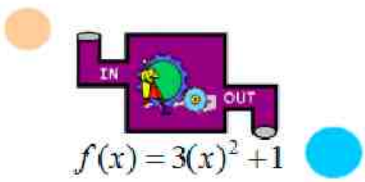
Example 2 (Cont.)

- Suppose you put [ ] you want into this machine, say 2.
- Then what would you get out?



$$f(x) = 3(x)^2 + 1$$

- Therefore $f(2) = 13$, which is a [ ]

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Example 2 (Cont.)

- We just showed that *f(x)* is a function if we are only considering {2} as our domain.
- The goal remember was to show that <u>all real numbers</u> are in the domain.
- Thus it will take us *forever* to show that <u>all real number</u> inputs give real number outputs.
- Therefore it is easier to start with the domain of formulas as *all real numbers* and then remove the 'bad' points.

'Bad' Points

- How do you know what the 'bad' points are when the function is given to you as a formula?
- These 'bad' points are those that don't have outputs for a selected input.
- The only way to not get real number outputs from formulas is to either;
  - Divide by 0 (which is undefined)
  - Or take the square root of negative numbers (which are no longer real numbers)

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Example 2 (Cont.)

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'Bad' Points

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- These are those that don't have outputs for a selected
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  - (which is undefined)
  - Or take the square root of (which are no longer real numbers)

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Example 2 (Cont.)

- And since $f(x) = 3x^2 + 1$ does not have a division sign or a square root symbol, there is no way to get any 'bad' points.
- Therefore *f(x)* **is** a well-defined *function* for all real numbers.

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Example 3

- Suppose $g(x) = \dfrac{2x+4}{x-1}$,
- Does $g(x)$ give a well-defined *function* for all real numbers?
  - i.e. Does $g(x)$ give real number outputs for all real number inputs?

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Example 2 (Cont.)

- And since $f(x) = 3x^2 + 1$ does [ ] have a division sign or a square root symbol, there is [ ] to get any 'bad' points.

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Example 3

- Suppose $g(x) = \dfrac{2x+4}{x-1}$,
- Does *g(x)* give a well-defined *function* for all real numbers?
  - i.e. Does g(x) give [ ] outputs for all [ ] inputs?

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Example 3 (Cont.)

- First off we notice that there is a division sign in this formula.
- That means that their will be 'bad' points that we will need to remove from the domain to make $g(x)$ a function.
- Again let's look to our function machine for help.

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Example 3 (Cont.)

- Suppose you put any real number you want into this machine, say 3.
- Then what would you get out?

$g(x) = \frac{2x+4}{x-1}$ $g(3) = \frac{2(3)+4}{3-1} = \frac{10}{2} =$ 5

- Therefore $g(3) = 5$, which is a real number.

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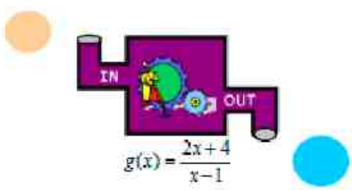
Example 3 (Cont.)

- First off we notice that there is a ____ in this formula.
- That means that their ____ 'bad' points that we will need to remove from the ____ to make $g(x)$ a function.
- Again let's look to our function ____ for help.

57

Example 3 (Cont.)

- Suppose you put ____ number you want into this machine, say 3.
- Then what would you get out?



- Therefore $g(3) =$ ____ which is a real number.

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Example 3 (Cont.)

- What if on the other hand you choose to put 1 into this machine.
- Then what would you get out?

$g(x)=\frac{2x+4}{x-1}$ $g(1)=\frac{2(1)+4}{1-1}=\frac{6}{0}$ Error

- Therefore since 1 does not have an output, because $g(1)$ is undefined, 1 cannot be in the domain of $g(x)$ and therefore must be removed.

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Example 3 (Cont.)

- This means that $g(x)=\frac{2x+4}{x-1}$

  does not give a well-defined *function* for all real numbers, because 1 is a real number but does not have an associated output value.

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Example 3 (Cont.)

- What if on the other hand you ________ to put 1 into this machine.
- Then what would you get out?

$g(x) = \frac{2x+4}{x-1}$

- Therefore since 1 ________ have an output, because $g(1)$ is undefined, 1 cannot be in the domain of $g(x)$ and therefore must be ________

59

Example 3 (Cont.)

- This means that $g(x) = \dfrac{2x+4}{x-1}$

____

, because 1 is a real number but does not have an associated output value.

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Example 4

- Similarly, you can simply ask what the domain must be in order to make an equation a function.
- What is the domain of $h(t) = \sqrt{2t+6}$?

Example 4 (Cont.)

- Since this equation has a square root symbol in it, we know that there are going to be 'bad' points.
- And as we said earlier, these 'bad' points occur when we take the square root of negative numbers.
- To avoid this we need to make sure that the only inputs we put into this function are those that make $2t + 6$ positive.

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Example 4

- Similarly, you can simply ask what the domain must be in order to make an ______ a function.
- What is the domain of $h(t) = \sqrt{2t+6}$?

61

Example 4 (Cont.)

- Since this equation has a ______ symbol in it, we know that there ______ going to be 'bad' points.
- And as we said earlier, these 'bad' points occur when we take the square root of ______ numbers.
- To avoid this we need to make sure that the ______ inputs we put into this function are those that make ______ positive.

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Example 4 (Cont.)

- That means that $2t + 6 \geq 0$.
- Then by solving for t we get, $t \geq -3$.
- Therefore the *domain* of $h(t)$ is, $\{t \mid t \geq -3\}$.

Question 10

- Does the formula $f(x) = \dfrac{2x}{3x+6}$ give a well-defined function for all real numbers?

- No, because $x = -2$ is a real number but there is no output value associated with it, because $f(-2)$ is undefined.

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Example 4 (Cont.)

- That means that $2t + 6 \geq$ ☐
- Then by solving for t we get, $t \geq -3$.
- Therefore the *domain* of $h(t)$ is, ☐.

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Question 10

- Does the formula $f(x) = \frac{2x}{3x+6}$ give a well-defined function for all real numbers?

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SUMMARY

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Relation (Definition)

- A *relation* is a correspondence between a first set, called the *domain*, and a second set, called the *range*, such that each member of the domain corresponds to *at least one* member of the range.

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SUMMARY

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Relation (Definition)

- A ________ is a correspondence between a ________ called the *domain*, and a second set, called the ________ such that each member of the domain corresponds to ________ *one* member of the range.

Function (Definition)

- A *function* is a type of relation in that it is a correspondence between a first set, called the *domain*, and a second set, called the *range*, however, each member of the domain must only correspond to *exactly one* member of the range.

Three Parts

- Domain
  - Set of Inputs
- Correspondence
  - Process linking the Inputs to the Outputs
- Range
  - Set of Outputs

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Function (Definition)

- A ________ is a type of relation in that it is a correspondence between a first set, called the ________ and a ________ called the *range*, however, each member of the domain must only correspond to ________ *one* member of the range.

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Three Parts

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Representations

- A *relation*, and therefore also a *function*, can be represented in several different ways
  - Table
  - Set of Points
  - Graphically
  - Defined by an Equation
  - Written Statement or Paragraph

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How to Tell if a Relation Is a Function?

- Three ways
  - Vertical line test
  - No input gets assigned to more than one output
  - Every input gets assigned to some output

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Representations

- A *relation*, and therefore also a ______ can be represented in several different ways
  -
  -
  -
  -
  -

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How to Tell if a Relation Is a Function?

- Three ways
  -
  -
  -

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ADDITIONAL RESOURCES

Including links to other helpful websites

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Function Graphers

- Basic Grapher
  - http://www.univie.ac.at/future.media/moe/fplotter/fplotter.html
- More Advanced Grapher
  - http://nlvm.usu.edu/en/nav/frames_asid_109_g_3_t_1.html?open=activities
- Quadratic Functions - changing coefficients
  - http://www.slu.edu/classes/maymk/Applets/FamilyGraphs.html

Presentation Only

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Linear Function Machine Games

- Basic
  - http://www.shodor.org/interactivate/activities/LinearFunctMachine/
- Pattern Recognition of only the range
  - i.e. you find the next output value
  - http://nlvm.usu.edu/en/nav/frames_asid_191_g_3_t_1.html
- More complete and harder function machine guesser
  - http://www.mathplayground.com/FunctionMachine.html
- Like the previous one, but with inverses
  - http://www.interactive-resources.co.uk/mathspack2/fmachine/function.html

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Function Lessons Online

- Function Lesson with Tables and Quiz
  - http://www.purplemath.com/modules/fcns.htm
- Overall Lesson on Functions including Machines
  - http://score.kings.k12.ca.us/lessons/functions/machine.html
- Great overall Math courses, with lectures, questions and quizzes
  - http://people.hofstra.edu/faculty/Stefan_Waner/Realworld/calctopic1/inverses.html#dandr

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Homework

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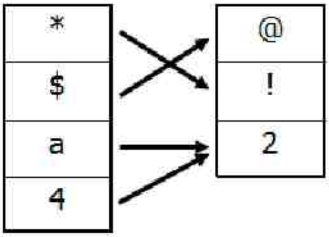
Exercise 1

- Does this table represent a function?



- What is the domain and range?

Domain =

Range =

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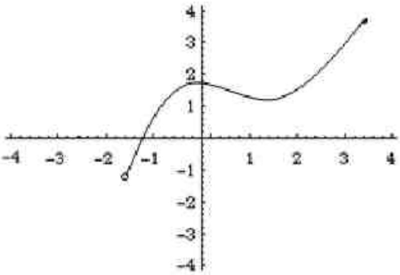
Exercise 3

- What is the domain and range of this function?

- Domain =
- Range =



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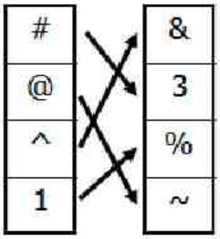
Exercise 2

- Do both of these represent the same function?

{(#,3), (@,%), (^,&), (1,~),}



- What is the domain and range?

Domain =

Range =

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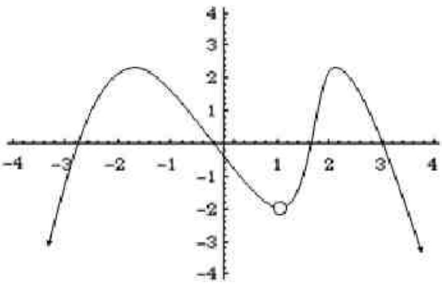
Exercise 4

- Does this graph represent a *function*?



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Homework

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Exercise 5

- Does this statement represent a function?
  - The correspondence that assigns to each actor the title of the movie that they were in.
- What is the Domain?
- What is the Range?

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Exercise 7

- Suppose $h(x) = \sqrt{x+2}$
- Does $h(x)$ give a well-defined *function* for all real numbers?

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Exercise 6

- How could you make that statement a function?

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Exercise 8

- What is the domain of $w(c) = \dfrac{3c+1}{c^2-4}$?
- Domain =

APPENDIX H

Interview Questions:

At the end of the semester, I asked two students who participated in the lectures on functions but never demonstrated understanding of the function concept to talk to me about where they were having difficulty answering the questions. I tried to talk as little as possible in hopes that the students would explain to me where they are having the difficulties. I kept a printout with me of the information below in order for me to spark their memory and also to limit my ability to help the students answer the questions. I only gave the students the first sheet. Then after we worked through that question, I gave them the second question to see if they could answer it on their own. These interviews will allow me to tailor future lectures on functions to emphasize and prioritize these missing pieces of their cognitive roots. I have included the questions I asked them and also the notes I kept with me for the interviews. (Interview)

Question

- Suppose $f(x)=\dfrac{x+5}{x-1}$, and $g(x)=\sqrt{x}$;
- Does $(f \circ g)(x)$ describe a well-defined function for *all real numbers*?
- Why or Why Not?
- What is the *domain* of $(f \circ g)(x)$?

First question and only thing the student had in front of them.

2^{nd} Question

- Suppose f(x) = , and g(x) = ;
- Does $(f \circ g)(x)$ describe a well-defined function for *all real numbers*?
- Why or Why Not?
- What is the *domain* of $(f \circ g)(x)$?

Second question was given after they completed the first one.

Below are my predetermined questions for the interview. I also showed the students the printout of the function machine to help them remember the Technology Lecture.

Question (Mine)

- Suppose $f(x)=\frac{x+5}{x-1}$, and $g(x)=\sqrt{x}$;
- Does $(f\circ g)(x)$ describe a well-defined function for *all real numbers*?

This is the goal

- Why or Why Not?
- I don't have a schedule
- Stay under 1 hour and don't skip any steps
- Where is the obstacle?
- Say only the minimum
- "Please explain to me what you are doing now"
- What is the *domain* of $(f\circ g)(x)$?
- **If they say they understand, have them pick two functions and then try this again.**

My Questions?

- Ask them to work on it and to ask questions about anything that they are having trouble with
- What part of the question are you having trouble with?
  - Our goal is to look at $(f\circ g)(x)$.
- Let's look at $f(x)$, do you know what that means?
  - If x is 2, then what is $f(2)$? $f(2)=\frac{(2)+5}{(2)-1}=\frac{7}{1}=7$
  - What is the domain of $f(x)$? $\{x \mid x \text{ is a real number, but } x\neq 1\}$
  - Would it help if I showed you a picture?
- What about $g(x)$?
  - If x is 4, what is $g(4)$? $g(4)=\sqrt{4}=2$
  - What is the domain of $g(x)$? $\{x \mid x\geq 0\}$
  - Would it help if I showed you a picture?
- What about $(f\circ g)(x)$?
  - Do you understand the notation?
    - It means $f(g(x))$
  - Is it confusing that both have x's in them?
  - Can you describe what is happening
  - Would it help if I showed you a picture of function composition?
    - Do you understand what is going on in the picture?
    - How does this function relate to the picture?
    - If x is 4, what is $(f\circ g)(4)$? $(f\circ g)(4)=f(g(4))=f(2)=7$
- Do you understand what well-defined means?
- Do you understand what all-real numbers mean?
- Do you know a function is?

Well-Defined

- What does well-defined for all real numbers mean?
  - It means that you get real number outputs for all real number inputs

- Do you think that $f(x)=\frac{x+5}{x-1}$ is well-defined for all real numbers? No
  - Why or Why not?

- What is the domain of $f(x)$? $\{x|x \text{ is a real number, but } x \neq 1\}$

- Does anything stand out now as to what well-defined for all real numbers means?

- What if I said that $f(x)=\frac{x+5}{x-1}$ is not well-defined for all real numbers, can you tell me why?

- What about $g(x)=\sqrt{x}$, is it well-defined for all real numbers? No
  - Why or why not?
  - What is the domain of g(x)?

$\{x|x \geq 0\}$

All Real Numbers

- What does all real numbers mean?
  - It means all the numbers on the number line. Like 2, 2 3 root2,…
- Is 2 a real number? Yes
- Is 5 3 a real number? Yes
- Is root2 a real number? Yes
- What is not a real number? 2i
- Is 2i a real number? No

$$(f \circ g)(x)$$

- Notation
  - What is (f0g)(x) short for? f(g(x))
  - How would you find (f0g)(4)?

$$(f \circ g)(4) = f(g(4)) = f(2) = 7$$

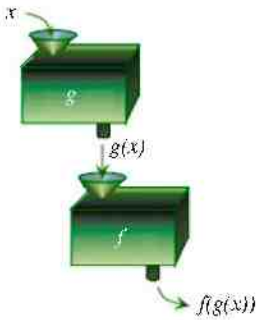
- Can you describe what is happening?

- What if I showed you a picture, would that help?

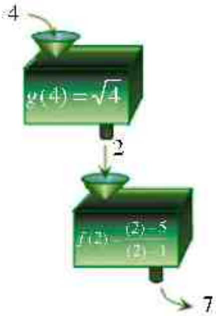
- Now can you find (g0f)(0)?

$$(g \circ f)(3) = g(f(3)) = g\left(\frac{(3)+5}{(3)-1}\right) = g\left(\frac{8}{2}\right) = g(4) = \sqrt{4} = 2$$

$$(f \circ g)(x)$$



$$(f \circ g)(4)$$



Domain

- What is the domain of a function?
  - It is the set of all the inputs to the function
  - All the stuff that gives you outputs

- How do you find the domain of a formula?
  - You start with all real numbers and then remove the 'bad points
  - The 'bad' points are those inputs that don't give you outputs
    - Divide by 0
    - Square root of negative numbers

- Example 1
  - What is the domain of $f(x)=\frac{x+5}{x-1}$?
    $\{x \mid x \text{ is a real number, but } x \neq 1\}$

- Example 2
  - What is the domain of $g(x)=\sqrt{x}$?
    $\{x \mid x \geq 0\}$

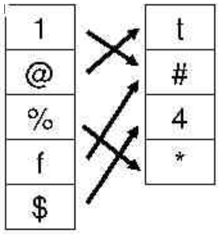
- You pick a function
  - What is the domain of it?

Function

- What is a function?
  - A correspondence between an input set, domain, and an output set, range, such that for each input there is only one possible output.

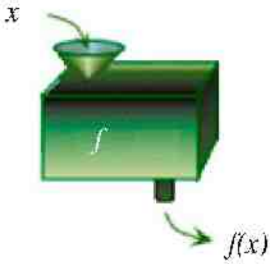
- How do you determine if something is a function?
  - Check all inputs to see if only get assigned to one output
  - Make sure there are no 'bad' points

- Example

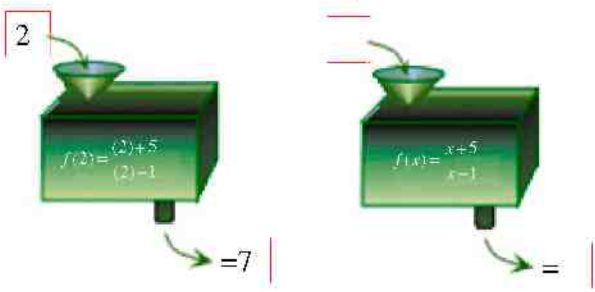


- What if I showed you a picture of a function machine, would that help?

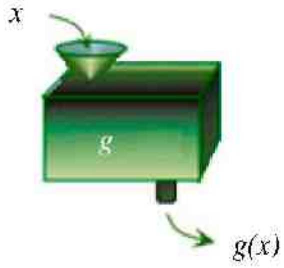
Function Machine for f



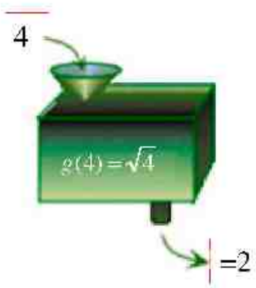
Example

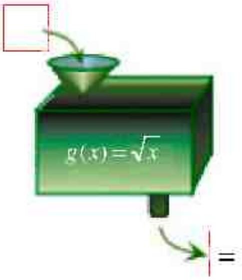


Function Machine for *g*



Example





I gave this to the students and asked them to fill in the blanks to come up with an example of a function machine and how it works.