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On the Distribution of Primes in an Imaginary Quadratic Number Ring

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Abstract

The study of prime numbers has been an area of interest in mathematics since antiquity. One natural question one may ask is "How many primes are there less than or equal to some positive integer?" The first attempts to answering this were in the late 1700s, culminating in the celebrated Prime Number Theorem. We investigate how this may be generalized to primes in an imaginary quadratic number rings in a given sector of the complex plane.

Contents

1. Introduction

1.1. **Historical Context.** Prime numbers have been of interest dating back to Euclid in the year 300 BC. In particular, he originally proved that there are infinitely many primes. Since then, much progress has been made in trying to understand the behavior of prime numbers.

The next natural question to consider is how many primes are there up to a given positive number x. For instance, one find that there are 25 prime numbers less than or equal to 100. Assuming that this pattern continues, one might then try to hypothesize that there must be 50 primes less than 200. This is not quite true, because the answer is actually 46.

The **Prime Number Theorem** allows us to accurately estimate $\pi(x)$, the number of primes less than or equal to some $x > 0$, by calculating the limit

$$\pi(x) \sim \frac{x}{\log x}\,;\text{ that is, } \lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1. \qquad (*)$$

Originally proved by Hadamard and de la Vallée Poussin in 1898 [1], this tells us that $\pi(x)$ is approximately equal to $\dfrac{x}{\log x}$ for sufficiently large x with diminishing error as $x \to \infty$. In fact, this theorem can be stated more strongly. In order to do this, we first give a brief introduction into the use of "Big O Notation".

Definition 1.1. We say that $f(x) = O(g(x))$ if there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all sufficiently large x.

As a consequence, $f(x) = g(x) + O(h(x))$ signifies that $(f - g)(x) = O(h(x))$. Now, we can state the Prime Number Theorem with O-term as given by de la Vallée Poussin in 1899 [21].

Theorem 1.2. (Prime Number Theorem) Let $\pi(x)$ denote the the number of primes less than or equal to some positive number x. Then, we have for some constant $a > 0$

$$\pi(x) = Li(x) + O(xe^{-a\sqrt{\log x}}),$$

where $Li(x) = \displaystyle\int_2^x \frac{dt}{\log t}$ denotes the logarithmic integral.

Integrating by parts on $Li(x)$ gives us the following weaker version of this theorem:

$$\pi(x) = \frac{x}{\log x} + O(xe^{-a\sqrt{\log x}}).$$

That is, we can think of $\frac{x}{\log x}$ as a good approximation for $\pi(x)$, whose error is at most $xe^{-a\sqrt{\log x}}$ for sufficiently large x. Dividing both sides by $\frac{x}{\log x}$ and letting $x \to \infty$ yields (*).

Similar statements to the Prime Number Theorem were proven for certain subsets of primes. In 1837, Dirichlet proved that for any given relatively prime integers a and d, there are infinitely many primes of the form $a + nd$. Moreover, there is a variant to the prime number theorem to this result [1]; if $\pi_{a,d}(x)$ denotes the number of primes of the form $a+nd$ less than or equal to x, then

$$\pi_{a,d}(x) \sim \frac{x}{\phi(d)\log x},$$

where $\phi(d)$ denotes the Euler ϕ-function.

In 1847, Gabriel Lamé tried to solve the ever-elusive Fermat's Last Theorem [23]. In doing so, his work had spawned the use of "adjoining values to the familiar integers." In other words, Lamé took the integers and added powers of an nth root of unity (giving what is today known as a ring of Cyclotomic integers). His work, along with other pioneers such as Dedekind and Dirichlet, led to the branch of mathematics known as Algebraic Number Theory [10].

We now review the pertinent concepts that will allow us to formulate analogues of the Prime Number Theorem and Dirichlet's Theorem that incorporate these "new" numbers.

1.2. **Terminology from Algebraic Number Theory.** We recall basic concepts and results from Algebraic Number Theory. Further background can be found in [20] and [23].

Definition 1.3. We say that a number α is an **algebraic number** (over $\mathbb{Q}$) if α is the zero of a polynomial with rational coefficients.

Given an algebraic number α, we consider the associated **algebraic number field** $K = \mathbb{Q}(\alpha)$, a finite field extension of $\mathbb{Q}$. Note that $i = \sqrt{-1}$ is an algebraic number, because i is a zero of $x^2 + 1$. This element gives rise to $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$, the field of Gaussian rational numbers.

Definition 1.4. An **algebraic integer** is a complex number that is the zero of a monic polynomial with integer coefficients.

Example 1.5. We see that $i, -i$ are algebraic integers since they are zeroes of the monic polynomial $x^2 + 1$. However, $\frac{1}{2}$ is not an algebraic integer, since any polynomial with integer coefficients having $\frac{1}{2}$ as a zero would have to have a factor of $2x - 1$ (thus making it not monic).

For any algebraic number field K, the set of algebraic integers in K forms a ring known as an **algebraic number ring**, denoted as $\mathcal{O}_K$ (or more briefly as $\mathcal{O}$ when K is understood). In particular, we are interested in algebraic number rings arising from a field extension of degree 2, i.e, those numbers α that are given by a polynomial of degree 2. Furthermore, we remark that $\mathbb{Q}$ is to $\mathbb{Z}$ as an algebraic number field K is to $\mathcal{O}$.

Definition 1.6. A field extension of degree 2 over $\mathbb{Q}$ is called a **quadratic field** and has the form $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \,:\, a, b \in \mathbb{Q}\}$, where d is a square-free integer. If $d < 0$, we say that it is an **imaginary quadratic field**.

The **quadratic number ring** $\mathcal{O}$ associated to $\mathbb{Q}(\sqrt{d})$ is the set of elements of the form $a + b\omega$, where $a, b \in \mathbb{Z}$ and

$$\omega = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \bmod 4 \\ \sqrt{d} & \text{otherwise.} \end{cases}$$

Example 1.7. The ring $\mathbb{Z}[i]$, whose elements are of the form $a + bi$, is commonly referred to as the Gaussian integers. Furthermore, letting $\zeta = \frac{-1+\sqrt{-3}}{2}$, we call $\mathbb{Z}[\zeta]$ the ring of Eisenstein integers.

Each quadratic field has the usual properties of a field, along with the operation of **conjugation**, that is for $\alpha = a + b\sqrt{d}$, we define $\overline{\alpha} = a - b\sqrt{d}$. We call the product of an element α and its conjugate $\overline{\alpha}$ the **norm**, denoted $N(\alpha) = \alpha\overline{\alpha}$.

Next, we discuss prime elements in an algebraic number ring.

Definition 1.8. We say that an element p is **prime** in $\mathcal{O}$ if whenever p divides ab for some $a, b \in \mathcal{O}$, then p divides a or p divides b. Furthermore, p must be non-zero and a non-unit.

Definition 1.9. We say that an element p is **irreducible** in $\mathcal{O}$ if it cannot be written as the product of two non-units in $\mathcal{O}$.

Although these definitions hold in $\mathbb{Z}$, this is not generally the case in an algebraic number ring. We give an example from [2] to highlight this concept.

Example 1.10. Consider $\mathbb{Z}[\sqrt{-5}]$, here we see that 6 factors as the product of $2 \cdot 3$, and also as $(1+\sqrt{-5})(1-\sqrt{-5})$. However, neither $2, 3$, nor $1 \pm \sqrt{-5}$, are units, and none are associates of one another.

It is known that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain. We note that in general, unique factorization into irreducible elements is not guaranteed in a quadratic number ring. In fact, there are only nine instances where an imaginary quadratic number ring possesses unique factorization! However, it turns out that we can bypass this difficulty by considering ideals.

Definition 1.11. We say that a proper ideal $\mathfrak{p}$ of a commutative ring R is a **prime ideal** if for $a, b \in R$ such that $ab \in \mathfrak{p}$, then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

In fact, if the algebraic number ring is a principal ideal domain (PID), then we have unique factorization into irreducible elements (see [23]). Otherwise, we content ourselves with the following key result that gives us an analogue for the Fundamental Theorem of Arithmetic, but for prime *ideals*.

Theorem 1.12. Every non-zero ideal of $\mathcal{O}$ can be written as a product of prime ideals, that are unique up to the order of the factors.

When working with ideals, we may also take the norm of an ideal $\mathfrak{a}$. This is computed by $\mathfrak{N}(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|$. For our purposes, we observe that the norm of an ideal satisfies two key properties:

- If $\mathfrak{a} = (\alpha)$ for some $\alpha \in \mathcal{O}$, then $\mathfrak{N}(\mathfrak{a}) = |N(\alpha)|$.
- For any $\mathfrak{a}, \mathfrak{b} \in \mathcal{O}$, we have $\mathfrak{N}(\mathfrak{a}\mathfrak{b}) = \mathfrak{N}(\mathfrak{a})\mathfrak{N}(\mathfrak{b})$.

From now on, unless otherwise explicitly stated, we are exclusively concerned with imaginary quadratic number rings.

The following is Edmund Landau's algebraic number theoretic extension of the Prime Number Theorem, known as the **Prime Ideal Theorem** [14].

Theorem 1.13. Let $\Pi_K(x)$ denote the number of prime ideals in a number field K with norm at most x. Then,

$$\Pi_K(x) = \sum_{\mathfrak{N}(\mathfrak{p}) \leq x} 1 = Li(x) + O(xe^{-\frac{b}{\sqrt{n}}\sqrt{\log x}}),$$

where $n = [K : \mathbb{Q}]$, and b is a positive constant independent of K.

Our goal is to prove an analogue of Dirichlet's Theorem in the case of an imaginary quadratic number ring that gives the number of prime ideals in a given sector in the complex plane. We start with a way to assign an angle to a prime ideal using Hecke's concept of a *prime ideal number* in the next section.

1.3. **Prime Ideal Numbers.** In [6], Erdös and Hall studied the angular distribution of Gaussian Integers with a fixed norm, which, geometrically, is the same as studying the distribution of points with integral coordinates in a circle. This notion can be extended to other quadratic extensions. In a general quadratic extension, the ring of integers might not be a principal ideal domain, but, Hecke introduced the concept of *ideal numbers* (see [8], [9]). This concept consists allows us to represent ideals with specific algebraic integers.

Definition 1.14. Given an imaginary quadratic number ring $\mathcal{O}$, suppose that its class group has order $h > 1$. Thus, we have a basis $B_1, B_2, ..., B_h$, with respective orders $c_1, c_2, ..., c_h$. In each class B_i, we choose one ideal $\mathfrak{b}_i$. By definition of a basis, every ideal is equivalent to a unique product $\mathfrak{b}_1^{m_1} \cdots \mathfrak{b}_k^{m_k}$ with $0 \leq m_i < c_i$. Therefore, for any ideal $\mathfrak{a}$, there exists $c \in K$ such that

$$\mathfrak{a} = c\mathfrak{b}_1^{m_1} \cdots \mathfrak{b}_h^{m_h}.$$

Now, $\mathfrak{b}_i^{c_i} = [\beta_i]$ for each $1 \leq i \leq k$, with $\beta_i \in \mathcal{O}_K$. Define $\omega_i = \sqrt[c_i]{\beta_i}$ where we fix some choice of c_i-th root of β_i. Then, $\alpha = c\omega_1^{m_1} \cdots \omega_k^{m_k}$ is called an **ideal number** associated to the ideal $\mathfrak{a}$.

Definition 1.15. We say that an ideal number α is **integral** if α is an algebraic integer. Furthermore, α is called a **prime ideal number** if the corresponding ideal is prime.

Example 1.16. Consider $K = \mathbb{Q}(\sqrt{-5})$ with $\mathcal{O} = \mathbb{Z}[\sqrt{-5}]$. We previously showed that $\mathcal{O}$ is not a UFD. In fact, it can be shown that $\mathcal{O}$ has class group $\{[1], [\langle 2, 1+\sqrt{-5}\rangle]\}$. Since $\langle 2, 1+\sqrt{-5}]\rangle^2 = \langle 2\rangle$, we can take $\sqrt{2}$ as a (prime) ideal number representing $[\langle 2, 1+\sqrt{-5}\rangle]$. Being the only nontrivial class, we conclude that any ideal $\mathfrak{a}$ has a corresponding ideal number α of the form $\alpha = c(\sqrt{2})^m$ for some $c \in \mathbb{Q}(\sqrt{-5})$ and $m \in \{0, 1\}$.

Remark: Using ideal numbers allows us to associate an angle to a given prime ideal, even when it is not principal.

Following ideas of Dias [5], we give a definition of how to associate an angle to an ideal.

Definition 1.17. Let $\mathfrak{p}$ be a prime ideal. Define the angle of the ideal $\mathfrak{p}$ as $\theta_{\mathfrak{p}} = \arg \alpha'$ (also written as $\arg \mathfrak{p}$), where α' is the unique associate of the prime ideal number α associated to $\mathfrak{p}$ satisfying $-\frac{\pi}{w} < \arg\ \alpha' \leq \frac{\pi}{w}$, where w denotes the number of units in $\mathcal{O}$.

Following [18], we generalize the prime ideal counting function.

Definition 1.18. Fix $-\frac{\pi}{w} < \phi_1 < \phi_2 \leq \frac{\pi}{w}$. Then we define

$$\Pi(x;\phi_1,\phi_2) = \sum_{\substack{\mathfrak{N}(\mathfrak{p})\leq x\\ \phi_1\leq\theta_\mathfrak{p}\leq\phi_2}} 1.$$

That is, $\Pi(x;\phi_1,\phi_2)$ denotes the number of prime ideals in a number field K with norm at most x and whose angle $\theta_\mathfrak{p}$ lies between ϕ_1 and ϕ_2, inclusive. With this function, we state the **Angular Prime Ideal Theorem** [18].

Theorem 1.19. We have

$$\Pi(x;\phi_1,\phi_2) = \frac{\phi_2-\phi_1}{2\pi}Li(x) + O(xe^{-c_{25}\sqrt{\log x}}).$$

We give the proof of this result in Section 2.4.

Remark: The result, like its predecessors, essentially follows from the non-vanishing of a special function in a specific region in the complex plane, in this case a Hecke L-function. We first define the specific Hecke L-function with which we prove this result and establish results culminating in its functional equation (analogous to that of the Riemann zeta function). We then establish non-vanishing results culminating in growth estimates to sums of Hecke characters. Finally, we employ Fourier Analysis to establish the desired Angular Prime Number Theorem.

2. The road to proving the Angular Prime Ideal Theorem

2.1. **Hecke L-function and its functional equation.** For the remainder of this thesis, we will assume that we are working in an *imaginary* quadratic number field.

All results in this section follow the methods of [18].

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

can be generalized to a ring of algebraic integers $\mathcal{O}$. This is called the **Dedekind zeta function** and is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where it is understood that we are summing over all nonzero ideals $\mathfrak{a}$ of $\mathcal{O}$. In the case where $\mathcal{O} = \mathbb{Z}$, we obtain the classic Riemann zeta function [20].

The Hecke L-function is a further generalization of the Dedekind zeta function. Now, we introduce **Hecke characters** (modulo (1)) for an imaginary quadratic ring. As its definition is rather involved, we fix the following notation. Let w denote the number of units of $\mathcal{O}$. In order to compute the Hecke character for a non-principal ideal, we use the ideal numbers introduced in the previous section.

Definition 2.1. Fix an integer a. Given an ideal $\mathfrak{a}$ with corresponding ideal number α, the Hecke character χ^{wa} of $\mathfrak{a}$ is defined as $\chi^{wa}(\mathfrak{a}) = \left(\frac{\alpha}{|\alpha|}\right)^{wa}$.

Remark: The Hecke character is a well-defined group homomorphism $\chi^{wa} : \mathcal{O} \to S^1$, because it returns the same value for any associate of μ. Moreover, for a principal ideal $\mathfrak{a} = (\mu)$ in $\mathcal{O}$, we have $\chi^{wa}(\mathfrak{a}) = \left(\frac{\mu}{|\mu|}\right)^{wa}$.

Definition 2.2. (Hecke L-function)

For $s \in \mathbb{C}$, we define $L(s, \chi^{wa}) = \displaystyle\sum_{\mathfrak{a}} \frac{\chi^{wa}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s}$, where $\mathfrak{a}$ varies over nonzero ideals of $\mathcal{O}$.

Remark: Like the Riemann and Dedekind zeta functions, the Hecke L-function has an analogous product expansion and region of convergence.

In the following theorem, we state where the series representation of the Hecke L-function converges.

Theorem 2.3. The L-series $L(s,\chi^{wa})$ is absolutely convergent when $\sigma > 1$ and uniformly convergent for $\sigma > 1+\delta$ for all $\delta > 0$. In particular, $L(s,\chi^{wa})$ is analytic for $\sigma > 1$.

Moreover, one has

$$L(s,\chi^{wa}) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1}, \tag{1}$$

where the product varies over the prime ideals $\mathfrak{p}$ of $\mathcal{O}$.

Proof: The idea for this proof comes from [18]. Convergence for the series representation for $L(s,\chi^{wa})$ directly follows upon comparing the absolute value of the series with the familiar p-series.

Next, we consider the convergence of the infinite product

$$\prod_{\mathfrak{p}} \left(1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1}.$$

By using the Maclaurin series for $\operatorname{Log}(1+z)$, we have

$$\left(1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1} = \exp\left(\sum_{m=1}^{\infty} \frac{\chi^{wa}(\mathfrak{p}^m)}{m\mathfrak{N}(\mathfrak{p})^{ms}}\right).$$

To show the convergence of the infinite product, observe that the absolute value of the latter expression $\left|\exp\left(\sum_{m=1}^{\infty} \frac{\chi^{wa}(\mathfrak{p}^m)}{m\mathfrak{N}(\mathfrak{p})^{ms}}\right)\right|$ is bounded above by

$$\exp\left(\sum_{m=1}^{\infty} \frac{1}{\mathfrak{N}(\mathfrak{p})^{m\sigma}}\right) = \exp\left(\frac{\mathfrak{N}(\mathfrak{p})^{-\sigma}}{1-\mathfrak{N}(\mathfrak{p})^{-\sigma}}\right) \le \exp(2\mathfrak{N}(\mathfrak{p})^{-\sigma}).$$

(The last inequality follows from $\mathfrak{N}(\mathfrak{p}) \geq 2$ for any prime ideal $\mathfrak{p}$.) Hence, the absolute value of the infinite product is bounded above by

$$\prod_{\mathfrak{p}} \exp\left(2\mathfrak{N}(\mathfrak{p})^{-\sigma}\right) = \exp\left(2\sum_{\mathfrak{p}} \mathfrak{N}(\mathfrak{p})^{-\sigma}\right) < \exp\left(2\sum_{\mathfrak{a}} \mathfrak{N}(\mathfrak{a})^{-\sigma}\right),$$

the latter of which converges for $\sigma > 1$ due to the presence of the Dedekind zeta function. Therefore, the infinite product converges absolutely for $\sigma > 1$.

Finally, we show that infinite series and product representations of the L-series are equal. To this end, note that for any $x > 0$, we have

$$\begin{aligned}
\prod_{\mathfrak{N}(\mathfrak{p}) \leq x} \left(1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1} &= \prod_{\mathfrak{N}(\mathfrak{p}) \leq x} \left(1 + \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s} + \frac{\chi^{wa}(\mathfrak{p})^2}{\mathfrak{N}(\mathfrak{p})^{2s}} + \cdots\right) \\
&= \sum_{\mathfrak{N}(\mathfrak{a}) \leq x}^{\star} \frac{\chi^{wa}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s} \\
&= \sum_{\mathfrak{a}} \frac{\chi^{wa}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s} + \sum_{\mathfrak{N}(\mathfrak{a}) > x}^{\star} \frac{\chi^{wa}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s},
\end{aligned}$$

where $\star$ denotes that we are summing over all nonzero ideals $\mathfrak{a}$ whose prime ideal factors each have norm at most x. In particular, letting $a = 0$ and $s = \sigma > 1$ in our result, we see that

$$\sum_{\mathfrak{N}(\mathfrak{a}) \leq x} \frac{1}{\mathfrak{N}(\mathfrak{a})^\sigma} < \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^\sigma}\right)^{-1}.$$

Thus, we have that $\sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^\sigma}$ converges. This in turn implies that

$$\left| \prod_{\mathfrak{N}(\mathfrak{p}) \leq x} \left(1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1} - \sum_{\mathfrak{N}(\mathfrak{a}) \leq x} \frac{\chi^{wa}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s} \right| \leq \sum_{\mathfrak{N}(\mathfrak{a}) > x}^{\star} \frac{1}{\mathfrak{N}(\mathfrak{a})^\sigma} \longrightarrow 0$$

as $x \to \infty$, thereby establishing the equality of the series and product representations. ■

The next theorem gives a series representation for the logarithmic derivative of the L-series.

Proposition 2.4. Let $s = \sigma + i\tau$. Then, when $\sigma > 1$

$$\frac{-L'(s,\chi^{wa})}{L(s,\chi^{wa})} = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\chi^{wa}(\mathfrak{p}^m) \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{ms}}.$$

Proof: By the previous theorem, we know that $L(s,\chi^{wa}) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1}$. Moreover, we established in the proof of the previous theorem that

$$\prod_{\mathfrak{p}} \left(1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1} = \prod_{\mathfrak{p}} \exp\left(\sum_{m=1}^{\infty} \frac{\chi^{wa}(\mathfrak{p}^m)}{m\mathfrak{N}(\mathfrak{p})^{ms}}\right).$$

Thus for any $\sigma > 1$, we have

$$\begin{aligned}
-\log L(s,\chi^{wa}) &= -\log \prod_{\mathfrak{p}} \left(1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1} \\
&= -\sum_{\mathfrak{p}} \log \left(1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1} \\
&= -\sum_{\mathfrak{p}} \log \exp\left(\sum_{m=1}^{\infty} \frac{\chi^{wa}(\mathfrak{p})^m}{m\mathfrak{N}(\mathfrak{p})^{ms}}\right) \\
&= -\sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\chi^{wa}(\mathfrak{p})^m}{m\mathfrak{N}(\mathfrak{p})^{ms}}.
\end{aligned}$$

Finally by applying logarithmic differentiation, we find that for $\sigma > 1$:

$$\frac{-L'(s,\chi^{wa})}{L(s,\chi^{wa})} = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\chi^{wa}(\mathfrak{p}^m) \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{ms}}. \quad \blacksquare$$

Now we introduce notation and terminology to define our version of the "Theta Function." Furthermore, we also provide tools and insight into better understanding the Hecke L-function. See [18].

Definition 2.5. A function $f \in C^{\infty}(\mathbb{R}^2)$ is called a *Schwartz function* if it approaches zero faster than any negative power of $x \in \mathbb{R}^2$ as $|x| \to \infty$, as do all of its derivatives.

We recall the traditional inner product.

Definition 2.6. ([18]) Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbb{R}^2$:

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2.$$

Then for a Schwartz function f, we define its Fourier transform $\hat{f}$ as

$$\hat{f}(y) = \int\limits_{\mathbb{R}^2} f(x) e^{2\pi i \langle x, y \rangle} \, dx.$$

We now give a brief definition of a quadratic form to be a homogeneous polynomial of degree two in a number of variables.

Remark: Let $Q(x)$ be a positive definite quadratic form given by a symmetric matrix A (that is, $Q(x) = \langle Ax, x \rangle$).

We have the following lemma.

Lemma 2.7. ([18], Lemma 9) Define $f_Q(x) = e^{-\pi Q(x)}$, where $Q(x)$ is a positive definite quadratic form. Then, $\hat{f}_Q(y) = \frac{1}{\sqrt{|A|}} f_{Q'}(y)$, where $Q'(x) = \langle A^{-1}x, x \rangle$.

Proof: Since A is positive definite, there exists a real matrix B such that $B^2 = A$. Then $Q(x) = \langle Ax, x \rangle = \langle Bx, Bx \rangle$, and thus we have $f_Q(x) = h_B(x)$, where $h(x) = e^{-\pi \langle x, x \rangle}$. Thus by our remark

$$\hat{f}_Q(y) = \hat{h}_B(y) = \frac{1}{|B|} \hat{h}(B^{-1}y) = \frac{1}{\sqrt{|A|}} h(B^{-1}y) = \frac{1}{\sqrt{|A|}} e^{-\pi Q'(y)}. \blacksquare$$

Now we introduce the Poisson summation formula. This equation allows us to rewrite Fourier coefficients of a function to the values of its Fourier transform.

Theorem 2.8. (Poisson Summation formula, [17]) Let f be a Schwartz function. Then for every $x \in \mathbb{R}^2$ we have

$$\sum_{m \in \mathbb{Z}^2} f(x + m) = \sum_{m \in \mathbb{Z}^2} \hat{f}(m) e^{2\pi i \langle m, x \rangle}.$$

Since $f_Q(x)$ is a Schwartz function, we can apply the Poisson summation formula to obtain the following abstraction of [18].

Theorem 2.9. Let $x_1, x_2 \in \mathbb{R}$ and define

$$\begin{cases} u_1 = x_1 + \omega x_2 \\ \\ u_2 = x_1 + \overline{\omega} x_2. \end{cases}$$

(Recall that $\{1, \omega\}$ is an integral basis for $\mathcal{O}$.) Then, for $t > 0$ we have the following formula:

$$\sum_{\mu \in \mathcal{O}} \exp\Big(-2\pi t(\mu + u_1)(\overline{\mu} + u_2)\Big) = \Big(\frac{1}{-it(\omega - \overline{\omega})}\Big) \sum_{\nu \in \mathcal{O}} \exp\Big(\frac{2\pi|\nu|^2}{t(\omega - \overline{\omega})^2} + \frac{2\pi i(-\nu u_1 + \overline{\nu} u_2)}{(\omega - \overline{\omega})}\Big).$$

Proof: Let $f_Q(x) = e^{-\pi Q(x)}$, where $Q(y)$ is the positive definite quadratic form

$$Q(y) = 2t|y_1 + y_2\omega|^2 = 2t\Big(y_1^2 + Tr(\omega)y_1y_2 + N(\omega)y_1^2\Big),$$

where $Tr(\omega) = \omega + \overline{\omega}$ and $N(\omega) = \omega\overline{\omega}$. Then $Q(y) = \langle Ay, y\rangle$, where

$$A = 2t \begin{bmatrix} 1 & \frac{1}{2}Tr(\omega) \\ \frac{1}{2}Tr(\omega) & N(\omega) \end{bmatrix}.$$

Since $|A| = 4t^2 N(\omega) - t^2 Tr(\omega)^2$, we have

$$A^{-1} = \frac{1}{4tN(\omega) - t\,Tr(\omega)^2} \begin{bmatrix} 2N(\omega) & -Tr(\omega) \\ -Tr(\omega) & 2 \end{bmatrix}.$$

Then, it follows that

$$Q'(y) = \langle A^{-1}y, y\rangle = \frac{1}{4tN(\omega) - tTr(\omega)^2}\Big(2N(\omega)y_1^2 - 2Tr(\omega)y_1y_2 + 2y_2^2\Big).$$

Now, consider

$$F(x) = \sum_{\mu \in \mathcal{O}} e^{-2\pi t(\mu+u_1)(\overline{\mu}+u_2)} = \sum_{\mu \in \mathcal{O}} e^{-2\pi t|\mu+u_1|^2}.$$

Observe that the second equality follows from $x_1, x_2 \in \mathbb{R}$ and $u_1 = \overline{u_2}$. Then, letting $\mu = m_1 + m_2\omega$, we obtain

$$F(x) = \sum_{m \in \mathbb{Z}^2} e^{-2\pi t|m_1+m_2\omega+x_1+x_2\omega|^2} = \sum_{m \in \mathbb{Z}^2} e^{-\pi Q(x+m)} = \sum_{m \in \mathbb{Z}^2} f_Q(x+m).$$

Using Poisson summation and Lemma 2.8, we get

$$F(x) = \sum_{m \in \mathbb{Z}^2} \hat{f}_Q(m) e^{2\pi i\langle m,x\rangle} = \frac{1}{\sqrt{|A|}} \sum_{m \in \mathbb{Z}^2} \exp(-\pi Q'(m) + 2\pi i\langle m,x\rangle).$$

Solving

$$\begin{cases} u_1 = x_1 + \omega x_2 \\ u_2 = x_1 + \overline{\omega} x_2 \end{cases}$$

for x_1, x_2, we get

$$\begin{cases} x_1 = \dfrac{\omega u_2 - \overline{\omega} u_1}{\omega - \overline{\omega}} \\ x_2 = \dfrac{u_1 - u_2}{\omega - \overline{\omega}}. \end{cases}$$

Therefore,

$$\begin{aligned} \langle m,x\rangle &= m_1\Big(\frac{\omega u_2 - \overline{\omega} u_1}{\omega - \overline{\omega}}\Big) + m_2\Big(\frac{u_1 - u_2}{\omega - \overline{\omega}}\Big) \\ &= \left(\frac{1}{\omega - \overline{\omega}}\right)\Big(\big(m_2 - m_1\overline{\omega}\big)u_1 + \big(m_1\omega - m_2\big)u_2\Big). \end{aligned}$$

To conclude this proof, set $\nu = m_1\overline{\omega} - m_2$. Then,

$$\langle m,x\rangle = \frac{1}{\omega - \overline{\omega}}\Big(\nu u_1 - \overline{\nu} u_2\Big).$$

Moreover,

$$Q'(m) = \frac{1}{4tN(\omega) - tTr(\omega)^2}\Big(2N(\omega)m_1^2 - 2Tr(\omega)m_1m_2 + 2m_2^2\Big)$$
$$= \frac{-2}{t(\omega-\overline{\omega})^2}\cdot|\nu|^2.$$

As m runs through $\mathbb{Z}^2$, ν runs through $\mathcal{O}$. Hence, we can conclude that

$$F(x) = \left(\frac{1}{-it(\omega-\overline{\omega})}\right)\sum_{\nu\in\mathcal{O}}\exp\left(\frac{2\pi|\nu|^2}{t(\omega-\overline{\omega})^2} + \frac{2\pi i}{(\omega-\overline{\omega})}(-\nu u_1 + \overline{\nu}u_2)\right). \blacksquare$$

The next two results extend the series from the previous theorem to $x_1, x_2 \in \mathbb{C}$.

We now give an abstraction of Proposition 11 from [18].

Theorem 2.10. The series

$$F(x) = \sum_{\mu\in\mathcal{O}} e^{-2\pi t(\mu+u_1)(\overline{\mu}+u_2)}$$

and

$$G(x) = \sum_{\nu\in\mathcal{O}}\exp\left(\frac{2\pi|\nu|^2}{t(\omega-\overline{\omega})^2} + \frac{2\pi i}{\omega-\overline{\omega}}\cdot(-\nu u_1 + \overline{\nu}u_2)\right)$$

are absolutely convergent for all $x = (x_1, x_2) \in \mathbb{C}^2$, and are uniformly convergent for all $R > 0$ in the region $\Omega_R = \{(x_1, x_2) \in \mathbb{C}^2 | \max\{|x_1|, |x_2|\} < R\}$.

Proof: We first establish the convergence properties for $F(x)$. Let $P_\mu(x) = 2\pi t(\mu+u_1)(\overline{\mu}+u_2)$ so that $F(x) = \sum_{\mu\in\mathcal{O}} e^{-P_\mu(x)}$. Since $P_\mu(x) = 2\pi t(|\mu|^2 + \mu u_2 + \overline{\mu}u_1 + u_1u_2)$, substituting $u_1 = x_1 + \omega x_2$ and $u_2 = x_1 + \overline{\omega}x_2$ into $P_\mu(x)$ yields

$$P_\mu(x) = 2\pi t\big(|\mu|^2 + (\mu+\overline{\mu})x_1 + (\mu\overline{\omega}+\overline{\mu}\omega)x_2 + x_1^2 + (\omega+\overline{\omega})x_1x_2 + |\omega|^2x_2^2\big).$$

Therefore in Ω_R, we have for all sufficiently large $|\mu|$:

$$\begin{aligned}\mathrm{Re}(P_\mu(x)) &> 2\pi t\big(|\mu|^2 - R|\mu+\overline{\mu}| - R|\mu\overline{\omega}+\overline{\mu}\omega| - R^2 - R^2|\omega+\overline{\omega}| - R^2|\omega|^2\big)\\ &= 2\pi t\big[|\mu|^2 - R(|\mu+\overline{\mu}| + |\mu\overline{\omega}+\overline{\mu}\omega|) - (1+|\omega+\overline{\omega}|+|\omega|^2)R^2\big]\\ &\geq C|\mu|^2.\end{aligned}$$

(Observe that the last inequality comes from any quadratic polynomial $q(t)$ is $\Theta(t^2)$.)

Now, it immediately follows that for all sufficiently large $|\mu|$, we have

$$|e^{-P_\mu(x)}| = e^{-\mathrm{Re}(P_\mu(x))} \leq e^{-C|\mu|^2} \leq e^{-C|\mu|}.$$

Since $\sum_{\mu\in\mathcal{O}} e^{-C|\mu|}$ is a convergent geometric series, we conclude for any $R > 0$ that $F(x)$ converges uniformly in Ω_R by the Weierstrass M-Test (and consequently is absolutely convergent for all $x \in \mathbb{C}^2$).

In a similar manner, one establishes the desired convergence properties for $G(x)$. ■

Next, we give a generalization of the Theorem 12 from [18], using very similar ideas.

Theorem 2.11. For $t > 0$ and arbitrary complex numbers u_1, u_2 we have

$$\sum_{\mu\in\mathcal{O}} e^{-2\pi t(\mu+u_1)(\overline{\mu}+u_2)} = \frac{1}{-it(\omega-\overline{\omega})}\sum_{\nu\in\mathcal{O}}\exp\Big(\frac{2\pi|\nu|^2}{t(\omega-\overline{\omega})^2} + \frac{2\pi i}{\omega-\overline{\omega}}\cdot(-\nu u_1 + \overline{\nu}u_2)\Big).$$

Proof: By Theorem 2.11, both sides of the equality in Theorem 2.10 define entire functions in $x_1, x_2 \in \mathbb{C}$. Since they are equal for *real* x_1, x_2, they must be equal by the Identity Theorem for analytic functions. ■

Next, we introduce our version Theta function for a number ring.

Definition 2.12. (Theta function) Define $\theta(t,a) = \sum\limits_{\mu\in\mathcal{O}} \mu^{wa}\exp(-\frac{2\pi i}{\omega-\overline{\omega}}\cdot t|\mu|^2)$.

Using the previous result, we now derive and prove the more general transformation formula for the Theta function.

Theorem 2.13. We have that $\theta(t,a)$ satisfies the equation $\theta(t,a) = t^{-1-wa} \cdot \theta(\frac{1}{t}, a)$.

Proof: Recall that

$$\sum_{\mu\in\mathcal{O}} \exp(-2\pi t(\mu+u_1)(\overline{\mu}+u_2)) = \frac{1}{-it(\omega-\overline{\omega})}\sum_{\nu\in\mathcal{O}} \exp\Big(\frac{2\pi|\nu|^2}{t(\omega-\overline{\omega})^2} + \frac{2\pi i}{\omega-\overline{\omega}}\cdot(-\nu u_1+\overline{\nu}u_2)\Big).$$

Let $\rho \in \mathcal{O}$ be arbitrary. Using the change of variables $u_1 = \rho$ and $u_2 = z + \overline{\rho}$ on the result to Theorem 2.11. Then, the left side of the last equality transforms as follows:

$$\sum_{\mu\in\mathcal{O}} \exp\Big(-2\pi t(\mu+u_1)(\overline{\mu}+u_2)\Big) = \sum_{\nu\in\mathcal{O}} \exp\Big(-2\pi t\big(|\mu+\rho|^2 + z(\mu+\rho)\big)\Big).$$

Moreover, the right side transforms as follows:

$$\frac{1}{-it(\omega-\overline{\omega})}\sum_{\nu\in\mathcal{O}} \exp\Big(\frac{2\pi|\nu|^2}{t(\omega-\overline{\omega})^2} + \frac{2\pi i}{\omega-\overline{\omega}}\cdot\big(-\nu\rho+\overline{\nu}(z+\overline{\rho})\big)\Big).$$

Taking our new representations of the left and right hand sides and differentiating $|wa|$ times with respect to z and simplifying, we obtain

$$\sum_{\mu\in\mathcal{O}} (\mu+\rho)^{wa} e^{-2\pi t\left(|\mu+\rho|^2+z(\mu+\rho)\right)} = \left(\frac{1}{-it(\omega-\overline{\omega})}\right)^{1+wa} \sum_{\nu\in\mathcal{O}} \overline{\nu}^{wa}\cdot e^{\frac{2\pi|\nu|^2}{t(\omega-\overline{\omega})^2}+\frac{2\pi i}{\omega-\overline{\omega}}\cdot\left(-\nu\rho+\overline{\nu}(z+\overline{\rho})\right)}.$$

Upon setting $z = 0$, our equation becomes

$$\sum_{\mu\in\mathcal{O}} (\mu+\rho)^{wa} e^{-2\pi t|\mu+\rho|^2} = \left(\frac{1}{-it(\omega-\overline{\omega})}\right)^{1+wa} \sum_{\nu\in\mathcal{O}} \overline{\nu}^{wa}\cdot e^{\frac{2\pi|\nu|^2}{t(\omega-\overline{\omega})^2}+\frac{2\pi i}{\omega-\overline{\omega}}\cdot(\overline{\nu\rho}-\nu\rho)}.$$

Furthermore, by setting $t = \frac{\tau}{-i(\omega-\overline{\omega})}$, the left hand side of our equation becomes

$$\sum_{\mu\in\mathcal{O}} (\mu+\rho)^{wa} e^{-2\pi t|\mu+\rho|^2} = \sum_{\mu\in\mathcal{O}} \mu^{wa}\cdot e^{\frac{-2\pi i}{(\omega-\overline{\omega})}\tau|\mu|^2} =: \theta(\tau,a).$$

Next, note that whether $-d \equiv 1 \mod 4$ or otherwise, $\frac{1}{(\omega-\overline{\omega})} \cdot (\overline{\nu\rho} - \nu\rho) = \frac{-2i}{\omega-\overline{\omega}} \operatorname{Im}(\nu\rho)$ is an integer; call it k. Then, $e^{2\pi i k} = 1$, rendering the corresponding factor in the sum above unnecessary. Thus, we have that

$$\begin{aligned}\theta(\tau, a) &= \Big(\frac{1}{\tau}\Big)^{1+wa} \cdot \sum_{\nu \in \mathcal{O}} \overline{\nu}^{wa} \cdot \exp\Big(\frac{-2\pi i |\nu|^2}{(\omega - \overline{\omega})\tau}\Big) \\ &= \tau^{-1-wa}\, \theta\Big(\frac{1}{\tau}, a\Big).\end{aligned}$$

The last equality follows from re-indexing the sum, replacing $\overline{\nu}$ with ν. ■

Next, we introduce and proof a functional equation for the L-function. We recall that a functional equation is an equation that specifies a function in implicitly form. We specifically use the functional equation for the Gamma Function, namely $\Gamma(x+1) = x\Gamma(x)$.

Abstracting yet another idea of [18], we give the following theorem.

Theorem 2.14. Let $\xi(s, \chi^{wa}) = \big(\frac{\omega-\overline{\omega}}{2\pi i}\big)^s \cdot \Gamma(s + \frac{w|a|}{2}) \cdot L(s, \chi^{wa})$. Then $\xi(s, \chi^{wa})$ is entire and satisfies $\xi(s, \chi^{wa}) = \xi(1-s, \chi^{wa})$.

Proof: Without loss of generality, we may assume that a is non-negative, since we have $L(s, \chi^{wa}) = L(s, \chi^{-wa})$. Fix $\mu \in K$ and $\sigma > 1$. By using the definition of the gamma function followed by the substitution $y = t|\mu|^2$, we have that

$$\begin{aligned}\Gamma\Big(s + \frac{w}{2}a\Big) \cdot |\mu|^{-2(s+\frac{wa}{2})} &= \int_0^\infty |\mu|^{-2(s+\frac{wa}{2})} \cdot y^{s+\frac{wa}{2}-1} \cdot e^{-y}\, dy \\ &= \int_0^\infty |\mu|^{-2(s+\frac{wa}{2})} (t|\mu|^2)^{s+\frac{wa}{2}-1} \cdot e^{-t|\mu|^2} \cdot (|\mu|^2\, dt) \\ &= \int_0^\infty t^{s+\frac{wa}{2}-1} e^{-t|\mu|^2}\, dt.\end{aligned}$$

Using this previous result, it follows that

$$\Gamma\left(s+\frac{w}{2}a\right)\frac{\chi^{wa}(\mu)}{\mathfrak{N}((\mu))^s}=\Gamma\left(s+\frac{w}{2}a\right)\cdot\frac{\mu^{wa}}{|\mu|^{2s+wa}}$$

$$=\left[|\mu|^{2(s+\frac{wa}{2})}\int\limits_0^\infty t^{s+\frac{wa}{2}-1}\cdot e^{-t|\mu|^2}dt\right]\cdot\frac{\mu^{wa}}{|\mu|^{2s+wa}}$$

$$=\int\limits_0^\infty \mu^{wa}\cdot t^{s+\frac{wa}{2}-1}\cdot e^{-t|\mu|^2}dt.$$

Next, by the change of variables $t=(\frac{2\pi i}{\omega-\overline{\omega}})\nu$ and $dt=(\frac{2\pi i}{\omega-\overline{\omega}})d\nu$, we have

$$\Gamma\left(s+\frac{w}{2}a\right)\frac{\chi^{wa}(\mu)}{\mathfrak{N}((\mu))^s}=\int\limits_0^\infty \mu^{wa}e^{-(\frac{2\pi i}{\omega-\overline{\omega}})\nu|\mu|^2}\cdot\left(\frac{2\pi i}{\omega-\overline{\omega}}\nu\right)^{s+\frac{wa}{2}-1}\cdot\left(\frac{2\pi i}{\omega-\overline{\omega}}\right)d\nu$$

$$=\left(\frac{2\pi i}{\omega-\overline{\omega}}\right)^{s+\frac{wa}{2}}\int\limits_0^\infty \mu^{wa}\cdot e^{-\frac{2\pi i}{\omega-\overline{\omega}}\nu|\mu|^2}\nu^{s+\frac{w}{2}a-1}d\nu.$$

Thus, we have

$$\Gamma\left(s+\frac{w}{2}a\right)\left(\frac{\omega-\overline{\omega}}{2\pi i}\right)^{s+\frac{wa}{2}-1}L(s,\chi^{wa})=\sum_{(\nu)\lhd\mathcal{O}}\int\limits_0^\infty \mu^{wa}\cdot e^{-\frac{2\pi i}{\omega-\overline{\omega}}\nu|\mu|^2}\nu^{s+\frac{wa}{2}-1}d\nu$$

$$=\frac{1}{w}\sum_{\nu\in\mathcal{O}}\int\limits_0^\infty \mu^{wa}\cdot e^{-\frac{2\pi i}{\omega-\overline{\omega}}\nu|\mu|^2}\nu^{s+\frac{wa}{2}-1}d\nu$$

$$=\frac{1}{w}\int\limits_0^\infty\sum_{\nu\in\mathcal{O}}\mu^{wa}\cdot e^{-\frac{2\pi i}{\omega-\overline{\omega}}\nu|\mu|^2}\nu^{s+\frac{wa}{2}-1}d\nu$$

$$=\frac{1}{w}\int\limits_0^\infty\theta(\nu,a)\nu^{s+\frac{wa}{2}-1}d\nu.$$

Next, we split up the integral and use the functional identity of the Theta function on the first term:

$$\Gamma\left(s+\frac{w}{2}a\right)\left(\frac{\omega-\overline{\omega}}{2\pi i}\right)^{s+\frac{wa}{2}}L(s,\chi^{wa}) = \frac{1}{w}\Big[\int_0^1 \theta(\nu,a)\nu^{s+\frac{wa}{2}-1} + \int_1^\infty \theta(\nu,a)\nu^{s+\frac{wa}{2}-1}d\nu\Big]$$

$$= \frac{1}{w}\Big[\int_0^1 \theta\Big(\frac{1}{\nu},a\Big)\nu^{s-\frac{wa}{2}-2}d\nu + \int_1^\infty \theta(\nu,a)\nu^{s+\frac{wa}{2}-1}d\nu\Big].$$

Applying the substitution $t=\frac{1}{\nu}$ and $dt=-\frac{1}{\nu^2}\,d\nu$ on the first integral yields

$$\int_0^1 \theta\Big(\frac{1}{\nu},a\Big)\nu^{s-\frac{wa}{2}-2}d\nu = \int_1^\infty \theta(t,a)t^{-s+\frac{wa}{2}}\,dt.$$

Therefore, we have that

$$\Gamma\left(s+\frac{w}{2}a\right)\left(\frac{\omega-\overline{\omega}}{2\pi i}\right)^{s+\frac{wa}{2}}L(s,\chi^{wa}) = \frac{1}{w}\int_1^\infty \theta(\nu,a)\left(\nu^{-s+\frac{wa}{2}}+\nu^{s+\frac{wa}{2}-1}\right)d\nu,$$

and we can conclude that

$$\xi(s,\chi^{wa}) = \frac{1}{w}\Big(\frac{\omega-\overline{\omega}}{2\pi i}\Big)^{-\frac{wa}{2}}\int_1^\infty \theta(\nu,a)\left(\nu^{s+\frac{wa}{2}-1}+\nu^{-s+\frac{wa}{2}}\right)d\nu.$$

Since this integral converges absolutely for all s, this representation is an analytic continuation of $\xi(s,\chi^{wa})$ to the whole plane. The functional equation then follows, since the right side of our result remains unchanged when we replace s by $1-s$. ■

We use the notation $\ll$ synonymously with our Big O notation. The following corollary follows from [18].

Corollary 2.15. In the strip $-\frac{1}{2}\leq\sigma\leq 4$ we have that $L(s,\chi^{wa})=k_1e^{k_2t}$ for some constants $k_1,k_2>0$ dependent only on a.

Proof: First, we use our definition of $\xi(s,\chi^{wa})$ and use our previous result to obtain the following equality.

$$\Big(\frac{\omega-\overline{\omega}}{2\pi i}\Big)^{s}\Gamma\Big(s+\frac{w|a|}{2}\Big)L(s,\chi^{wa})=\frac{1}{w}\Big(\frac{\omega-\overline{\omega}}{2\pi i}\Big)^{-\frac{wa}{2}}\int\limits_{1}^{\infty}\theta(\nu,a)\Big(\nu^{s+\frac{wa}{2}-1}+\nu^{-s+\frac{wa}{2}}\Big)d\nu$$

By combining terms we see that

$$\begin{aligned}\Big(\frac{\omega-\overline{\omega}}{2\pi i}\Big)^{s+\frac{wa}{2}}\Gamma\Big(s+\frac{w|a|}{2}\Big)L(s,\chi^{wa})&=\frac{1}{w}\int_{1}^{\infty}\theta(\nu,a)(\nu^{s+\frac{wa}{2}-1}+\nu^{-s+\frac{wa}{2}})\,d\nu\\&\ll\int_{1}^{\infty}\exp\Big(\frac{-2\pi iu}{\omega-\overline{\omega}}\Big)u^{4+\frac{wa}{2}-1}\,du\\&\ll O_a(1),\end{aligned}$$

due to the convergence of the integral for the Gamma function. Hence,

$$L(s,\chi^{wa})\ll_a\Big(\frac{2\pi i}{\omega-\overline{\omega}}\Big)^{s+\frac{wa}{2}}\frac{1}{\Gamma(s+\frac{|wa|}{2})}=O_a\Big(\frac{1}{|\Gamma(s+\frac{|wa|}{2})|}\Big).$$

Next, we apply Stirling's formula, which states that in an angular region $-\pi+\delta<\arg s<\pi+\delta$ for any fixed $\delta>0$, we have as $|s|\to\infty$

$$\log\Gamma(s)=\Big(s-\frac{1}{2}\Big)\log s-s+\frac{1}{2}\log 2\pi+O(|s|^{-1}).$$

This implies that

$$\log\frac{1}{\Gamma(s)}=\Big(\frac{1}{2}-s\Big)\log s+s+O(1).$$

Since $-\frac{\pi}{2}<\arg(s+|\frac{wa}{2}|)<\frac{\pi}{2}$, it follows that in the strip $-\frac{1}{2}\leq\delta\leq 4$, we have

$$\begin{aligned}\mathrm{Re}\Big(\log\frac{1}{\Gamma(s+|\frac{wa}{2}|)}\Big)&=\frac{1}{2}\log\Big|s+|\frac{wa}{2}|\Big|-\Big(\sigma-|\frac{wa}{2}|\Big)\log\Big|s+|\frac{wa}{2}|\Big|\\&\quad+t\arg\Big(s+\Big|\frac{wa}{2}\Big|\Big)+\sigma+\Big|\frac{wa}{2}\Big|+O(1)\\&<k_2|t|\end{aligned}$$

for some constant $k_2 > 0$. The corollary now follows from exponentiation both sides. ■

2.2. **Finding zero-free regions of the Hecke L-function.** We begin by citing two lemmas from Landau in order to help find zero-free regions for the Hecke L-functions $L(s, \chi^{wa})$.

Lemma 2.16. (Landau [16]) Let $r > 0$ be constant. Suppose that $f(s)$ is analytic for $|s - s_0| \leq r$. Furthermore, suppose

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M \; for \; |s - s_0| \leq r$$

and

$$f(s) \neq 0 \text{ whenever } |s - s_0| \leq r \text{ and } \mathrm{Re}(s) > \mathrm{Re}(s_0).$$

Then the following holds:

(1)

$$-\mathrm{Re}\left(\frac{f'(s_0)}{f(s_0)}\right) < \frac{4M}{r}.$$

(2) If there is a zero ρ on the line between $s_0 - \frac{r}{2}$ and s_0 (exclusive), then

$$-\mathrm{Re}\left(\frac{f'(s_0)}{f(s_0)}\right) < \frac{4M}{r} - \frac{1}{s_0 - \rho}.$$

Lemma 2.17. (Landau [16]) Let $r > 0$. Suppose that $f(s)$ is analytic for $|s - s_0| \leq r$ and on this region satisfies $\mathrm{Re}(f(s)) \leq M$ for some constant $M > 0$. Then for $|s - s_0| \leq \rho$ where $0 < \rho < r$, we have

$$|f'(s)| \leq \frac{2r}{(r - \rho)^2}(|M| + |f(s_0)|).$$

Next we cite the Phragmèn - Lindelöf Theorem on a half strip [3], an extension of the Maximum Modulus Principle to unbounded regions in the complex planes. This allows us to show that our L-function is bounded in the preceding proof.

Theorem 2.18. (Phragmèn - Lindelöf) Let f be a holomorphic function on the horizontal half-strip $\{z = \sigma + it : a \leq \sigma \leq b \text{ and } t \geq t_0 \geq 0\}$ with fixed a, b, t_0.

Suppose that for some $\alpha \geq 1$, we have $f(\sigma + it) = O(e^{t^\alpha})$ for all $t \geq t_0$, and on the sides of the half-strip f is bounded. Then, f is bounded on the half-strip.

Our next generalization from [18] gives a bound on the Hecke L-Function.

Theorem 2.19. In the strip $-\frac{1}{2} \leq \sigma \leq 4$, we have that $|L(s, \chi^{wa})| < c_1(1 + |a|)^2(1 + |t|)^2$ for some constant $c_1 > 0$ independent of a.

Remark: We note that the labelling $c_1, c_2, ...$ will be used to denote arbitrary, but different constants.

Proof: From the functional equation for the L-function, we have

$$\left|L\left(-\frac{1}{2} + it, \chi^{wa}\right)\right| = \left|\left(\frac{\omega - \overline{\omega}}{2\pi i}\right)^{2-2it}\right| \cdot \left|\frac{\Gamma(\frac{3}{2} - it + \frac{w}{2}|a|)}{\Gamma(-\frac{1}{2} + it + \frac{w}{2}|a|)}\right| \cdot \left|L\left(\frac{3}{2} - it, \chi^{wa}\right)\right|.$$

First we note that $|L(\frac{3}{2} - it, \chi^{wa})| \leq \sum\limits_{(\mu)} \frac{1}{N(\mu)^{3/2}} = O(1)$ due to p-series convergence. Thus, we obtain

$$\left|L\left(-\tfrac{1}{2} + it, \chi^{wa}\right)\right| \leq c_2 \left|\tfrac{\Gamma(\frac{3}{2} - it + \frac{w}{2}|a|)}{\Gamma(-\frac{1}{2} + it + \frac{w}{2}|a|)}\right|$$

for some constant $c_2 > 0$. By applying the functional equation for the gamma function, we obtain

$$\begin{aligned}
\left|L(-\frac{1}{2} + it, \chi^{wa})\right| &\leq c_2 \left|\frac{\Gamma(\frac{3}{2} - it + \frac{w}{2}|a|)}{\Gamma(-\frac{1}{2} + it + \frac{w}{2}|a|)}\right| \\
&= c_2 \left|\frac{(\frac{1}{2} - it + \frac{w}{2}|a|)(-\frac{1}{2} - it + \frac{w}{2}|a|)\Gamma(-\frac{1}{2} - it + \frac{w}{2}|a|)}{\Gamma(-\frac{1}{2} + it + \frac{w}{2}|a|)}\right| \\
&= c_2 \left|\frac{1}{2} - it + \frac{w}{2}|a|\right| \left| -\frac{1}{2} - it + \frac{w}{2}|a|\right|.
\end{aligned}$$

Further simplification gives us

$$
\begin{aligned}
\left|L(-\frac{1}{2}+it,\chi^{wa})\right| &= \frac{c_2}{4}\sqrt{(1+w|a|)^2+4t^2}\cdot\sqrt{(-1+w|a|)^2+4t^2} \\
&\leq \frac{c_2}{4}\Big((1+w|a|)+2|t|\Big)\Big((-1+w|a|)+2|t|\Big) \\
&< \frac{c_2}{4}\Big((w|a|+w|a|)+2|t|\Big)\Big((0+w|a|)+2|t|\Big) \\
&= \frac{c_2}{2}\Big(w^2|a|^2+2w|at|+2|t|^2\Big) \\
&\leq \frac{Kc_2}{2}\Big(|a|^2+|at|+|t|^2\Big) \text{ where } K=\max\{2,w^2,2w\} \\
&< \frac{Kc_2}{2}\Big(|a|^2+|at|+|t|^2+1\Big)^2 \\
&= c_3(1+|a|)^2(1+|t|)^2, \text{ where } c_3=\frac{Kc_2}{2}.
\end{aligned}
$$

Furthermore, $|L(4+it,\chi^{wa})| \leq \sum_{(\mu)} \frac{1}{N(\mu)^4} = O(1)$ again due to p-series convergence. Thus we obtain a similar result for $L(4+it,\chi^{wa})$. Now consider the function

$$\Lambda(s) = \frac{L(s,\chi^{wa})}{(1+|a|)^2(1+s)^2}.$$

As $\Lambda(s)$ is holomorphic in the strip $-\frac{1}{2} \leq \sigma \leq 4$, and since $|\Lambda(s)| \ll \frac{|L(s,\chi^{wa})|}{(1+|a|)^2(1+|t|)^2}$, it follows that $\Lambda(s)$ is bounded on $\sigma = -\frac{1}{2}$ and $\sigma = 4$ by above. Furthermore, it is $O(e^{ct})$ in the whole strip by Corollary 2.16. Thus, by Phragmén-Lindelöf's Theorem, we have that since our holomorphic function $\Lambda(s)$ is bounded the whole strip, and the theorem follows. ■

The following Lemma is a generalization of Lemma 19 from [18].

Lemma 2.20. In the strip $1 < \sigma < 2$,

$$\zeta_K(s) < \frac{2\phi(D)}{\sigma - 1},$$

where D is the discriminant of K.

Proof: First of all, observe that we can factor $\zeta_K(s)$ as follows:

$$\begin{aligned}
\zeta_K(s) &= \prod_{\mathfrak{p}} (1 - \mathfrak{N}(\mathfrak{p})^{-s})^{-1} \\
&= \prod_{p \text{ ramifies}} (1 - p^{-s})^{-1} \prod_{p \text{ splits}} (1 - p^{-s})^{-2} \prod_{p \text{ is inert}} (1 - p^{-2s})^{-1} \\
&= \prod_{p} (1 - p^{-s})^{-1} \prod_{p \text{ splits}} (1 - p^{-s})^{-1} \prod_{p \text{ is inert}} (1 + p^{-s})^{-1} \\
&= \zeta(s) L(s, \chi).
\end{aligned}$$

On the previous line, $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ is the Dirichlet L-Function for the χ-character $\chi(n) = \left(\frac{D}{n}\right)$, being the Kronecker extension of the Legendre/Jacobi symbol.

To conclude the proof, it remains to bound both $\zeta(s)$ and $L(s,\chi)$. First of all, since $\sigma > 1$, we have

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} n^{-\sigma} \leq 1 + \int_1^{\infty} u^{-\sigma} du = 1 + \frac{1}{\sigma - 1} < \frac{2}{\sigma - 1}.$$

Furthermore, to bound $L(s,\chi)$, we employ summation by parts:

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \int_1^{\infty} u^{-s} d\{\sum_{n \leq u} \chi(n)\} = s \int_1^{\infty} u^{-s-1} \big(\sum_{n \leq u} \chi(n)\big) du.$$

Since $|\sum_{n \leq x} \chi(n)| \leq \phi(D)$ for all $x \geq 1$, by [11], we conclude that $|L(s,\chi)| \leq \phi(D)$, and the lemma then immediately follows. ■

Now we give our general theorem that defines so called "Zero-Free Regions" for the Hecke L-Functions. The benefit to having such regions allows the possibility of having $\frac{1}{\log x}$ in our results without the need to mention domain restrictions.

Theorem 2.21. There exist constants $c_5, c_6 > 0$ such that $L(s, \chi^{wa})$ has no zeros in the region defined by

$$\sigma \geq \begin{cases} 1 - \frac{1}{c_5 \log\left((1+|a|)(1+|t|)\right)} & \text{for } |t| \geq c_6 \\ 1 - \frac{1}{c_5 \log\left((1+|a|)(1+|c_6|)\right)} & \text{for } |t| \leq c_6. \end{cases}$$

Proof: Logarithmic differentiation of (1) yields

$$-\frac{L'(s,\chi^{wa})}{L(s,\chi^{wa})} = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\chi^{wma}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{ms}}$$

for $\sigma > 1$. We recall the definition of $\theta_{\mathfrak{p}}$ as the unique angle in $(-\frac{\pi}{w}, \frac{\pi}{w}]$. In terms of $\theta_{\mathfrak{p}}$ we have, for every $\mathfrak{p}$ and m,

$$\frac{\chi^{wma}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{ms}} = \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{m\sigma}} \exp\Big(i(wma\theta_{\mathfrak{p}} - mt \log \mathfrak{N}(\mathfrak{p}))\Big).$$

Using the inequality $3 + 4\cos\phi + \cos(2\phi) = 2(1+\cos\phi)^2 \geq 0$, we deduce that

$$\mathrm{Re}\Big[-3\frac{\zeta_K'(\sigma)}{\zeta_K(\sigma)} - 4\left(\frac{L'(\sigma+it,\chi^{wa})}{L(\sigma+it,\chi^{wa})}\right) - \left(\frac{L'(\sigma+2it,\chi^{2wa})}{L(\sigma+2it,\chi^{2wa})}\right)\Big] \tag{2}$$

$$= \sum_{\mathfrak{p}} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{m\sigma}} \Big(3+4\cos\left(wma\theta_{\mathfrak{p}} - mt\log\mathfrak{N}(\mathfrak{p})\right)$$

$$+ \cos(2wma\theta_{\mathfrak{p}} - 2mt \log \mathfrak{N}(\mathfrak{p}))\Big) \geq 0.$$

Now, let $s_0 = \rho + i\tau$, where $1 < \rho < 2$ with ρ being suitably chosen as a function of τ. In the disk $|s - s_0| \leq \frac{3}{2}$ we have, by Corollary 2.20 and Lemma 2.21

$$\begin{aligned}\left|\frac{L(s,\chi^{wa})}{L(s_0,\chi^{wa})}\right| &< c_1(1+|a|)^2(1+|t|)^2\left|L(s_0,\chi^{wa})^{-1}\right| \\ &= c_1(1+|a|)^2(1+|t|)^2\left|\sum_{(\alpha)}\frac{\mu(\alpha)\chi^{wa}(\alpha)}{N(\alpha)^{s_0}}\right| \\ &\leq c_1(1+|a|)^2(1+|t|)^2\zeta_K(\rho) \\ &< \frac{c_7}{\rho-1}(1+|a|)^2(1+|t|)^2.\end{aligned}$$

Suppose now that $\mu + i\tau$ is a zero of $L(s,\chi^{wa})$, where $\rho - \frac{3}{4} < \mu < \rho$. Then, by applying part 2 of Lemma 4.1 with $r = \frac{3}{2}$ and

$$M = \log\left(\frac{c_7}{\rho-1}(1+|a|)^2(1+|\tau|)^2\right),$$

we find that

$$-\operatorname{Re}\left(\frac{L'(\rho+i\tau,\chi^{wa})}{L(\rho+i\tau,\chi^{wa})}\right) < \frac{16}{3}\log((1+|a|)(1+|\tau|) - \frac{8}{3}\log(\rho-1) + c_8. \tag{3}$$

In regards to $\frac{\zeta_K'}{\zeta_K}$, we note that because of the simple pole at 1,

$$-\frac{\zeta_K'(\rho)}{\zeta_K(\rho)} < \frac{1}{\rho-1} + c_{10}. \tag{4}$$

Now (2), (3), and (4) imply

$$\frac{4}{\rho-\mu} < \frac{3}{\rho-1} + \frac{80}{3}\log\big((1+|a|)(1+|\tau|)\big) - \frac{40}{3}\log(\rho-1) + c_{11}. \tag{5}$$

We may choose c_6 sufficiently large to ensure that for $|\tau| \geq c_6$

$$40\log(100\log(2(1+|\tau|)) + 3c_{11} < 20\log(2(1+|\tau|)),$$

and for all $a \neq 0$

$$40\log\Big(100\log\big((1+|a|)(1+|\tau|)\big)\Big) + 3c_{11} < 20\log\big((1+|a|)(1+|\tau|)\big). \qquad (6)$$

Now we set

$$\rho = \begin{cases} 1 + \frac{1}{100\log\big((1+|a|)(1+|\tau|\big)} & \text{for } |t| \geq c_6 \\ 1 - \frac{1}{100\log\big((1+|a|)(1+|\tau|\big)} & \text{for } |t| < c_6. \end{cases}$$

First, suppose that $\tau \geq c_6$. Put $\mathcal{L} = \log\big((1+|a|)(1+|\tau|)\big)$. Then (5), multiplied by 3 becomes

$$\frac{12}{\rho - \mu} < 980\mathcal{L} + 40\log(100\mathcal{L}) + 3c_{11}.$$

Thus, by (6) we have that $\frac{12}{\rho-\mu} < 1000\mathcal{L}$, and hence

$$\mu < 1 + \tfrac{1}{100\mathcal{L}} - \tfrac{12}{1000\mathcal{L}} = 1 - \tfrac{1}{500\mathcal{L}}$$

for all eventual zeros $\mu + i\tau$.

Next, suppose that $|\tau| < c_6$ and put $\mathcal{L}' = \log\big((1+|a|)(1+|c_6|)\big)$. In a similar manner, we find that

$$\mu < 1 - \frac{1}{500\mathcal{L}'},$$

thereby finishing the proof of this theorem.

Lemma 2.22. On the line $\sigma = 2$, we have $|\log L(s, \chi^{wa})| < c_{16}$ for some constant $c_{16} > 0$.

Proof: Writing $s = 2 + it$ for some $t \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{1}{|L(s, \chi^{wa})|} &= \prod_{\mathfrak{p}} \left| 1 - \frac{\chi^{wa}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s} \right| \\
&\leq \prod_{\mathfrak{p}} (1 + \mathfrak{N}(\mathfrak{p})^{-2}) \\
&\leq 1 + \sum_{\mathfrak{a}} \mathfrak{N}(\mathfrak{a})^{-2} \\
&= 1 + \zeta_K(2).
\end{aligned}$$

Moreover, since $|\chi^{wa}(\mathfrak{a})| = 1$ for any nonzero ideal $\mathfrak{a}$ in $\mathcal{O}$, we have

$$|L(s, \chi^{wa})| \leq \sum_{\mathfrak{a}} \left| \frac{\chi^{wa}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^2} \right| = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^2} = \zeta_K(2).$$

Thus, $\frac{1}{1+\zeta_K(2)} \leq |L(s, \chi^{wa})| \leq \zeta_K(2)$, and we conclude that the logarithm of the L-function is bounded as well. ■

The ideas for the proof of following theorem were taken from ([18], Theorem 22).

Theorem 2.23. In the region Ω defined by

$$3 \geq \sigma \geq \begin{cases} 1 - \frac{1}{c_{12} \log\left((1+|a|)(1+|t|)\right)} & \text{for } |t| \geq c_6 \\ 1 - \frac{1}{c_{12} \log\left((1+|a|)(1+|c_6|)\right)} & \text{for } |t| \leq c_6 \end{cases}$$

where $c_{12} > c_5$, we have

$$\left| \frac{L'(s, \chi^{wa})}{L(s, \chi^{wa})} \right| \leq c_{13} \log^3 \Big((1 + |a|)(1 + \max(|t_0|, c_6)) \Big).$$

Proof: Let $c_{14} > c_5$. For every $s_0 = 2 + it_0$ on the line $\mathrm{Re}(s) = 2$, let $\mathcal{C}_{s_0}$ be the circle with center at s_0 and radius

$$r = \begin{cases} 1 + \frac{1}{c_{14}\log\big((1+|a|)(1+|t_0|)\big)} & \text{for } |t| \geq c_6 \\ 1 + \frac{1}{c_{14}\log\big((1+|a|)(1+|c_6|)\big)} & \text{for } |t| \leq c_6. \end{cases}$$

We may freely assume that $c_6 > \frac{1}{2}$, so that $(1+|a|)(1+c_6) > 3$. Then we have for $s = \sigma + it$ in $\mathcal{C}_{s_0}$:

$$\log\big((1+|a|)(1+|t|)\big) \leq \log\big((1+|a|)(3+|t_0|)\big)$$

$$\leq \log\big((1+|a|)(1+|t_0|)\big) + \log 3.$$

Thus we obtain

$$\log\big((1+|a|)(1+|t|)\big) < \begin{cases} 2\log\big((1+|a|)(1+|t_0|)\big) \text{ for } |t_0| \geq c_6 \\ 2\log\big((1+|a|)(1+|c_6|)\big) \text{ for } |t_0| \leq c_6. \end{cases} \tag{7}$$

Now we use Lemma 2.18 with $f(s) = \log L(s, \chi^{wa})$, which is analytic in the zero-free region of Theorem 2.22, and thus analytic in $\mathcal{C}_{s_0}$.

Also, by Theorem 2.20

$$|L(s, \chi^{wa})| < c_1(1+|a|)^2(1+|t|)^2$$

in $\mathcal{C}_{s_0}$, so by (7)

$$\mathrm{Re}(f(s)) = \mathrm{Re}(\log(L(s, \chi^{wa})))$$

$$< 4\log\big((1+|a|)(1+|t|) + \log c_1$$

$$< 4\log\Big((1+|a|)\big(1+\max(|t_0|, c_6)\big)\Big) + c_{15}.$$

Now, we set

$$\rho = \begin{cases} 1 + \frac{1}{c_{12} \log\left((1+|a|)(1+|t_0|)\right)} & \text{for } |t_0| \geq c_6 \\ 1 + \frac{1}{c_{12} \log\left((1+|a|)(1+|c_6|)\right)} & \text{for } |t_0| \leq c_6, \end{cases}$$

where $c_{12} > c_{14}$. Then, Lemma 2.17 with $M = 4\log\left((1+|a|)(1+\max(|t_0|, c_6))\right) + c_{15}$, and Lemma 2.23 imply that in the disk $|s - s_0| \leq \rho$, we have

$$\begin{aligned} \left|\frac{L'(s,\chi^{wa})}{L(s,\chi^{wa})}\right| &< \frac{2r}{(r-\rho)^2}\left(4\log\left((1+|a|)\big(1+\max(|t_0|, c_6)\big)\right) + c_{15} + |f(s_0)|\right) \\ &= \frac{2r}{(r-\rho)^2}\left(4\log\left((1+|a|)\big(1+\max(|t_0|, c_6)\big)\right) + c_{16}\right), \end{aligned}$$

where $c_{16} = c_{15} + |f(s_0)|$. However,

$$\begin{aligned} \frac{2r}{(r-\rho)^2} &= \frac{2r}{(\frac{1}{c_{14}} - \frac{1}{c_{12}})^2} \log^2\left((1+|a|)\big(1+\max(|t_0|, c_6)\big)\right) \\ &\leq \frac{4}{(\frac{1}{c_{14}} - \frac{1}{c_{12}})^2} \log^2\left((1+|a|)\big(1+\max(|t_0|, c_6)\big)\right). \end{aligned}$$

Now, the statement of the theorem follows. ■

2.3. **Growth estimates for sums of Hecke characters.** In the following section, we find growth estimates on the sums of Hecke characters.

Definition 2.24. Let $K(s,\chi^{wa}) = \sum_{\mathfrak{p}} \frac{\chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}$, for any $\sigma > 1$.

It can be readily checked that this series is analytic for $\sigma > 1$, being absolutely convergent for $\sigma > 1$ and uniformly convergent for $\sigma \geq 1 + \delta$ for any $\delta > 0$. Moreover, we have the following generalizing result.

Lemma 2.25. $K(s,\chi^{wa})$ is analytic in the region Ω of Theorem 2.24 and satisfies in Ω

$$|K(s,\chi^{wa})| \leq \begin{cases} c_{15}\log^3\big((1+|a|)(1+|t|)\big) \text{ if } |t| \geq c_6 \\ c_{15}\log^3\big((1+|a|)(1+|c_6|)\big) \text{ if } |t| \leq c_6. \end{cases}$$

Proof: By Proposition 2.4 we have for $\sigma > 1$

$$K(s,\chi^{wa}) = -\frac{L'(s,\chi^{wa})}{L(s,\chi^{wa})} - \sum_{\mathfrak{p}}\sum_{m=2}^{\infty}\frac{\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{ms}}. \tag{8}$$

However, the logarithmic derivative is analytic in Ω, and the sum on the right is absolutely convergent for $\sigma > \frac{1}{2}$ and uniformly convergent for $\sigma \geq \frac{1}{2}+\delta$ for any $\delta > 0$, because

$$\begin{aligned}\sum_{\mathfrak{p}}\sum_{m=2}^{\infty}\frac{\log\mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{m(\frac{1}{2}+\sigma)}} &= \sum_{\mathfrak{p}}\frac{\log\mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{\frac{1}{2}+\sigma}(\mathfrak{N}(\mathfrak{p})^{\frac{1}{2}+\delta}-1)} \\ &\leq 2\sum_{\mathfrak{p}}\frac{\log p^2}{p^{\frac{1}{2}+\delta}(p^{\frac{1}{2}+\delta}-1)} \\ &< 4\sum_{n=2}^{\infty}\frac{\log n}{n^{\frac{1}{2}+\delta}(n^{\frac{1}{2}+\delta}-1)} \\ &< 4\sum_{n=2}^{\infty}\frac{\log n}{n^{1+2\delta}},\end{aligned}$$

and this later sum is seen to be convergent upon comparison with $\sum\frac{1}{n^{1+2\delta-\epsilon}}$, where we choose ϵ such that $0 < \epsilon < 2\delta$ (note that we are using the fact that for any constant $c > 0$, we have $\log x < x^c$ for all sufficiently large x).

Then, (8) constitutes an analytic continuation of $K(s,\chi^{wa})$ to Ω (since Ω lies to the right of the line $\sigma = \frac{1}{2}$). From (8), it is also clear that in Ω,

$$\left|K(s,\chi^{wa}) + \frac{L'(s,\chi^{wa})}{L(s,\chi^{wa})}\right| < c_{17}$$

for some constant c_{17}. Therefore, in Ω we have

$$|K(s,\chi^{wa})| < c_{13}\log^3\Big((1+|a|)(1+\max(|t_0|,c_6))\Big) + c_{17}$$
$$< c_{15}\log^3\Big((1+|a|)(1+\max(|t_0|,c_6))\Big). \qquad \blacksquare$$

Lemma 2.26. ([18], Lemma 24)

$$\int_{2-i\infty}^{2+i\infty} \frac{x^s}{s^2}ds = \begin{cases} 0 & if\ 0 < x < 1 \\ 2\pi i \log x & if\ x \geq 1 \end{cases}$$

Proof: Let $x > 0$ be fixed. The function $\frac{x^2}{s^2}$ is analytic in the whole plane, except for a double pole in the point $s = 0$ with residue $\log x$. Assume first that $0 < x < 1$. Using the integration contour of Figure 1, we have by Cauchy's Theorem

$$\int_{2-iR}^{2+iR} \frac{x^s}{s^2}ds + \int_{\gamma_R} \frac{x^s}{s^2}ds = 0.$$

However,

$$\left|\int_{\gamma_R} \frac{x^s}{s^2}ds\right| \leq \pi\frac{x^2}{R^s},$$

so by letting $R \to \infty$ we get

$$\int_{2-i\infty}^{2+i\infty} \frac{x^s}{s^2}ds = 0.$$

Next, assume that $x \geq 1$. We then use the contour of Figure 2. Since the pole lies inside the contour, by Cauchy's Theorem we get that

$$\int_{2-iR}^{2+iR} \frac{x^s}{s^2}ds + \int_{\omega_R} \frac{x^s}{s^2}ds = 2\pi i \log x.$$

However,

$$\left|\int_{\omega_R} \frac{x^s}{s^2}ds\right| \leq \pi\frac{x^s}{R^s},$$

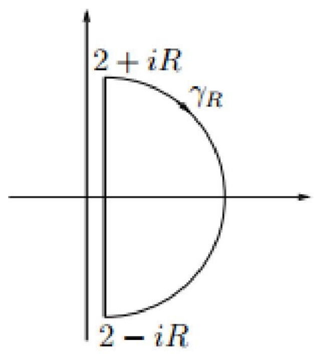

Figure 1

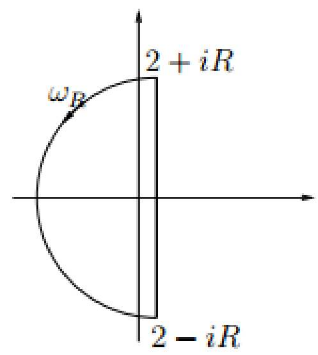

Figure 2

so again, letting $R \to \infty$, we see that

$$\int_{2-i\infty}^{2+i\infty} \frac{x^s}{s^2} ds = 2\pi i \log x. \qquad \blacksquare$$

The next generalization of Theorem 25 in [18], allows us to compute sums of prime ideals.

Theorem 2.27. For $x > 1$

$$\sum_{\mathfrak{N}(\mathfrak{p}) \leq x} \chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p}) \log \frac{x}{\mathfrak{N}(\mathfrak{p})} \ll x e^{-c_{18} \frac{\log x}{\log(1+|a|)+\sqrt{\log x}}} \log^3(1+|a|).$$

Proof: By Lemma 2.27,

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s}{s^2} K(s, \chi^{wa}) ds = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s}{s^2} \left(\frac{\chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s} \right) ds \tag{9}$$

The latter integral simplifies as follows:

$$\sum_{\mathfrak{p}} \chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p}) \frac{1}{2\pi i} \left(\int_{2-i\infty}^{2+i\infty} \frac{\left(\frac{x}{\mathfrak{N}(\mathfrak{p})}\right)^2}{s^2} ds \right)$$

$$= \sum_{\mathfrak{N}(\mathfrak{p}) \leq x} \chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p}) \log \frac{x}{\mathfrak{N}(\mathfrak{p})},$$

which is the sum we want to approximate. Now let ω be the curve defined by

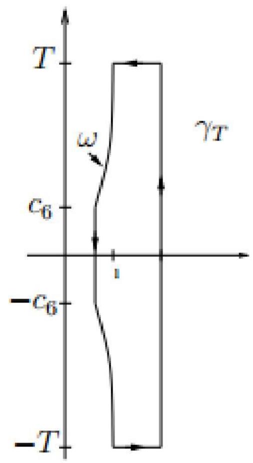

Figure 3

$$\sigma = \begin{cases} 1 - \frac{1}{c_{12}\log\left((1+|a|)(1+|t|)\right)} & for\ |t| \geq c_6 \\ 1 - \frac{1}{c_{12}\log\left((1+|a|)(1+|c_6|)\right)} & for\ |t| \leq c_6. \end{cases}$$

We claim that

$$\int_{2-i\infty}^{2+i\infty} \frac{x^s}{s^2} K(s, \chi^{wa}) ds = \int_{\omega} \frac{x^s}{s^2} K(s, \chi^{wa})\ ds. \tag{10}$$

To see this, consider for large T the contour Γ_T defined in the following manner:

- From $2 - iT$ to $2 + iT$ in a straight line,
- From $2 + iT$ to $1 - \frac{1}{c_{12}\mathcal{L}} + iT$ in a straight line,
- From $1 - \frac{1}{c_{12}\mathcal{L}} + iT$ to $1 - \frac{1}{c_{12}\mathcal{L}} - iT$ along ω,
- From $1 - \frac{1}{c_{12}\mathcal{L}} - it$ to $2 - iT$ in a straight line.

For the aforementioned contour, we are letting $\mathcal{L} = \log\big((1 + |a|)(1 + |T|)\big)$.

Since the integrand is analytic inside and on Γ_T, by Cauchy's Theorem,

$$0 = \int\limits_{\Gamma_T} \frac{x^s}{s^2} K(s, \chi^{wa}) ds = \left(\int\limits_{2+iT}^{2-iT} + \int\limits_{2+iT}^{1-\frac{1}{c_{12}\mathcal{L}}+iT} + \int\limits_{1-\frac{1}{c_{12}\mathcal{L}}+iT}^{1-\frac{1}{c_{12}\mathcal{L}}-iT} + \int\limits_{1-\frac{1}{c_{12}\mathcal{L}}-iT}^{2-iT} \right) \frac{x^s}{s^2} K(s, \chi^{wa}) ds. \tag{11}$$

By Lemma 2.26,

$$\left| \int\limits_{1-\frac{1}{c_{12}\mathcal{L}}\pm iT}^{2\pm iT} \frac{x^s}{s^2} K(s, \chi^{wa}) ds \right| \leq \frac{x^s}{T^s} c_{15} \log^3 \big((1+|a|)(1+T)\big),$$

which goes to 0 as $T \to \infty$ in (11). Hence, the horizontal integrals vanish, implying (11). Thus we need to approximate the integral along ω. Now, for an arbitrary $\tau > c_6$, we have by Lemma 2.26

$$\int\limits_{\omega} \frac{x^s}{s^2} K(s, \chi^{wa}) ds \ll \int\limits_{0}^{c_6} \frac{x^{1-\frac{1}{c_{12}\log((1+|a|)(1+c_6))}}}{1+t^2} \log^3((1+|a|)(1+c^6))\, dt$$
$$+ \left(\int\limits_{c_6}^{\tau} + \int\limits_{\tau}^{\infty} \right) \frac{x^{1-\frac{1}{c_{12}\log((1+|a|)(1+t))}}}{t^2} \log^3((1+|a|)(1+t))\, dt.$$

The upper bound for the right side can be further simplified to

$$x^{1-\frac{1}{c_{12}\log((1+|a|)(1+c_6))}} \log^3(1+|a|)$$
$$+ x^{1-\frac{1}{c_{12}\log((1+|a|)(1+\tau))}} \int\limits_{1}^{\infty} \frac{\log^3\big((1+|a|)(1+t)\big)}{t^2} dt$$
$$+ x \int\limits_{\tau}^{\infty} \frac{\log^3\big((1+|a|)(1+t)\big)}{t^2} dt.$$

Putting together the first two terms in the previous line, we may further asymptotically simplify to

$$x e^{-\frac{\log x}{c_{12}\log((1+|a|)(1+\tau))}} \log^3(1+|a|)$$
$$+ \frac{x}{\tau} \log^3 \tau \log^3(1+|a|).$$

By setting $\tau = e^{\sqrt{\log x}}$, we get

$$\int_{\omega} \frac{x^s}{s^2} K(s,\chi^{wa}) ds \ll x \log^3(1+|a|)\left(e^{-c_{19}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}} + e^{\frac{3}{2}\log\log x - \sqrt{\log x}}\right).$$

However, for sufficiently large x,

$$\frac{3}{2}\log\log x - \sqrt{\log x} < -c_{20}\sqrt{\log x} < -c_{20}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}},$$

so letting $c_{18} = \min(c_{19}, c_{20})$ and recalling (9) and (10), we deduce that

$$\sum_{\mathfrak{N}(\mathfrak{p})\le x} \chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p}) \log \frac{x}{\mathfrak{N}(\mathfrak{p})} \ll x e^{-c_{18}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}} \log^3(1+|a|). \qquad \blacksquare$$

Theorem 2.28. For $x > 1$, we have

$$\sum_{\mathfrak{N}(\mathfrak{p})\le x} \chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p}) \ll x e^{-c_{21}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}} \log^3(1+|a|).$$

Proof: Set for short $\delta = \delta(x) = e^{-\frac{1}{2}c_{18}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}}$. With x replaced by $(1+\delta)(x)$, Theorem 2.28 gives

$$\sum_{\mathfrak{N}(\mathfrak{p})\le (1+\delta)x} \chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p}) \log \frac{(1+\delta)x}{\mathfrak{N}(\mathfrak{p})} \ll (1+\delta)x e^{-c_{18}\frac{\log(1+\delta)x}{\log(1+|a|)+\sqrt{\log(1+\delta)x}}} \log^3(1+|a|) \ll \delta^2 x \log^3(1+|a|). \tag{12}$$

We will now split the sum on the left in two parts. First, again using Theorem 2.28, we have

$$\sum_{\mathfrak{N}(\mathfrak{p})\le x} \chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p}) \log \frac{(1+\delta)x}{\mathfrak{N}(\mathfrak{p})} = \sum_{\mathfrak{N}(\mathfrak{p})\le x} \chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p}) \left(\log \frac{(1+\delta)}{\mathfrak{N}(\mathfrak{p})} + \log \frac{x}{\mathfrak{N}(\mathfrak{p})}\right) \tag{13}$$

Upon splitting the terms in the product, the right side of the previous equality becomes

$$\sum_{\mathfrak{N}(\mathfrak{p})\le x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})\log\frac{(1+\delta)}{\mathfrak{N}(\mathfrak{p})}+\sum_{\mathfrak{N}(\mathfrak{p})\le x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})\log\frac{x}{\mathfrak{N}(\mathfrak{p})}$$

$$=\log(1+\delta)\sum_{\mathfrak{N}(\mathfrak{p})\le x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})+\sum_{\mathfrak{N}(\mathfrak{p})\le x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})\log\frac{x}{\mathfrak{N}(\mathfrak{p})}$$

$$=\log(1+\delta)\sum_{\mathfrak{N}(\mathfrak{p})\le x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})+O\big(\delta^2x\log^3(1+|a|)\big).$$

Secondly,

$$\sum_{x<\mathfrak{N}(\mathfrak{p})\le(1+\delta)x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})\log\frac{(1+\delta)x}{\mathfrak{N}(\mathfrak{p})}\ll\delta x\log((1+\delta)x)\log(1+\delta)$$

$$\ll\delta^2x\log x,$$

since the number of terms in this sum is $O(\delta x)$. By (13), we have

$$\log(1+\delta)\sum_{\mathfrak{N}(\mathfrak{p})\le x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})=\sum_{\mathfrak{N}(\mathfrak{p})\le x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})\log\frac{(1+\delta)x}{\mathfrak{N}(\mathfrak{p})}$$

$$+O\big(\delta^2x\log^3(1+|a|)\big)$$

$$=\sum_{\mathfrak{N}(\mathfrak{p})\le(1+\delta)x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})\log\frac{(1+\delta)x}{\mathfrak{N}(\mathfrak{p})}$$

$$-\sum_{x<\mathfrak{N}(\mathfrak{p})\le(1+\delta)x}\chi^{wa}(\mathfrak{p})\log\mathfrak{N}(\mathfrak{p})\log\frac{(1+\delta)x}{\mathfrak{N}(\mathfrak{p})}$$

$$+O\big(\delta^2x\log^3(1+|a|)\big).$$

Remark, label our last equation as (14).

From (12) and (13), we note that

$$\delta^2x\log^3(1+|a|)+\delta^2x\log x+\delta^2x\log^3(1+|a|)$$

$$=O\big(\delta^2x\log x\log^3(1+|a|)\big).$$

Therefore, we can rewrite (??) as

$$\sum_{\mathfrak{N}(\mathfrak{p})\leq x} \chi^{wa} \log \mathfrak{N}(\mathfrak{p}) \ll \frac{\delta^2}{\log(1+\delta)} x \log x \log^3(1+|a|)$$

$$\ll \delta x \log x \log^3(1+|a|)$$

$$= x e^{-\frac{1}{2}c_{18}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}+\log\log x} \log^3(1+|a|)$$

$$= x e^{c_{21}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}} \log^3(1+|a|). \qquad \blacksquare$$

We now give a final growth estimate.

Theorem 2.29. For $x > 1$, we have

$$\sum_{\mathfrak{N}(\mathfrak{p})\leq x} \chi^{wa}(\mathfrak{p}) \ll x e^{-c_{21}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}} \log^3(1+|a|).$$

Proof: Let

$$\vartheta(x) = \sum_{\mathfrak{N}(\mathfrak{p})\leq x} \chi^{wa}(\mathfrak{p}) \log \mathfrak{N}(\mathfrak{p}).$$

By partial summation, we obtain

$$\sum_{\mathfrak{N}(\mathfrak{p})\leq x} \chi^{wa}(\mathfrak{p}) = \sum_{2\leq m\leq x} \frac{\vartheta(m)-\vartheta(m-1)}{\log m}$$

$$= \sum_{2\leq m\leq x} \vartheta(m)\left(\frac{1}{\log m} - \frac{1}{\log(m+1)}\right) + \frac{\vartheta(x)}{\log([x]+1)}.$$

Then Theorem 2.29 yields

$$\sum_{\mathfrak{N}(\mathfrak{p})\leq x} \chi^{wa}(\mathfrak{p}) \ll \sum_{2\leq m\leq x} m e^{-c_{21}\frac{\log m}{\log(1+|a|)+\sqrt{\log m}}} \log^3(1+|a|)\left(\frac{1}{\log m} - \frac{1}{\log(m+1)}\right)$$

$$+ x e^{-c_{21}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}} \log^3(1+|a|).$$

Since $x e^{-c_{21}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}}$ is monotone and increasing for sufficiently large x,

$$
\begin{aligned}
&\sum_{2\le m\le x} me^{-c_{21}\frac{\log m}{\log(1+|a|)+\sqrt{\log m}}}\log^3(1+|a|)\left(\frac{1}{\log m}-\frac{1}{\log(m+1)}\right)\\
&\ll xe^{-c_{21}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}}\sum_{2\le m\le x}\left(\frac{1}{\log m}-\frac{1}{\log(m+1)}\right)\\
&\ll xe^{-c_{21}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}}\left(\frac{1}{\log 2}-\frac{1}{\log([x]+1)}\right)\\
&\ll xe^{-c_{21}\frac{\log x}{\log(1+|a|)+\sqrt{\log x}}}\log^3(1+|a|).
\end{aligned}
$$

This completes the proof of Theorem 2.30.

2.4. **Proof of the Angular Prime Ideal Theorem.** We are almost ready to prove the Angular Prime Ideal Theorem. The final ingredient to prove this is a little Fourier analysis. More specifically, we apply a lemma of Vinogradov that will prove useful in the preceding results. The two lemmas construct two periodic functions which attain the value 0 on a particular interval, 1 on another interval, and some values between 0 and 1 exclusively elsewhere. By creating these two functions, we find the upper and lower bounds for our function $\Pi(x;\phi_1,\phi_2)$ that gives us the desired result.

Lemma 2.30. (Vinogradov, [18])

Let r be a positive integer, and let α, β, and Δ be real numbers satisfying

$$0<\Delta<1 \text{ and } \Delta\le\beta-\alpha\le 1-\Delta.$$

Then there exists a periodic function $\psi(x)$, with period 1 satisfying

(1) $\psi(x)=1$ in the interval $\alpha+\frac{\Delta}{2}\le x\le\beta-\frac{\Delta}{2}$,

(2) $\psi(x)=0$ in the interval $\beta+\frac{\Delta}{2}\le x\le 1+\alpha-\frac{\Delta}{2}$,

(3) $0\le\psi(x)\le 1$ in the remainder of the interval $\alpha-\frac{\Delta}{2}\le x\le 1+\alpha-\frac{\Delta}{2}$,

(4) $\psi(x)$ has an expansion in Fourier series of the form

$$\psi(x) = (\beta - \alpha) + \sum_{m=1}^{\infty} \Big(a_m \cos(2\pi m x) + b_m \sin(2\pi m x) \Big),$$

where

$$|a_m|, |b_m| \leq 2(\pi m)^{-1},$$

$$|a_m|, |b_m| < \left(\frac{2}{\pi m}\right)\left(\frac{r}{\pi m \Delta}\right)^r.$$

Lemma 2.31. Let $\delta > 0$ and suppose $2\delta \leq \phi_2 - \phi_1 \leq 2\pi - 2\delta$. Then there exists 2π-periodic functions $\overline{f}(\phi)$ and $\underline{f}(\phi)$ such that

(1) $\overline{f}(\phi) = 1$ if $\phi_1 \leq \phi \leq \phi_2$

$\overline{f}(\phi) = 0$ if $\phi_2 + \delta \leq \phi \leq 2\pi + \phi_1 - \delta$

$0 \leq \overline{f}(\phi) \leq 1$ in the rest of the interval $\phi_1 - \delta \leq \phi \leq 2\pi + \phi_1 - \delta$.

(2) $\underline{f}(\phi) = 1$ if $\phi_1 + \delta \leq \phi \leq \phi_2 - \delta$

$\underline{f}(\phi) = 0$ if $\phi_2 \leq \phi \leq 2\pi + \phi_1$

$0 \leq \underline{f}(\phi) \leq 1$ in the rest of the interval $\phi_1 \leq \phi \leq 2\pi + \phi_1$.

(3) If

$$\overline{f}(\phi) = \sum_{n=-\infty}^{\infty} \overline{a}_n e^{in\phi}, \underline{f}(\phi) = \sum_{n=-\infty}^{\infty} \underline{a}_n e^{in\phi}$$

then we have

$$\overline{a}_0 = \frac{1}{2\pi}(\phi_2 - \phi_1 + \delta), \qquad \overline{a}_n \ll \frac{1}{|n|}, \qquad \overline{a}_n \ll \frac{1}{\delta |n|^2}$$

$$\underline{a}_0 = \frac{1}{2\pi}(\phi_2 - \phi_1 - \delta), \qquad \underline{a}_n \ll \frac{1}{|n|}, \qquad \underline{a}_n \ll \frac{1}{\delta |n|^2}.$$

Proof: This follows directly from Lemma 6.4 if we take $x = \frac{w}{2\pi}\phi$ and $r = 1$, setting for $\overline{f}$,

$$\alpha = \frac{1}{2\pi}\phi_1 - \frac{1}{2\pi}\delta, \qquad \beta = \frac{1}{4\pi}\phi_2 + \frac{1}{2\pi}\delta, \qquad \Delta = \frac{1}{2\pi}\delta,$$

and for $\underline{f}$

$$\alpha = \frac{1}{2\pi}\phi_1 + \frac{1}{2\pi}\delta, \qquad \beta = \frac{1}{4\pi}\phi_2 - \frac{1}{2\pi}\delta, \qquad \Delta = \frac{1}{2\pi}\delta. \quad \blacksquare$$

Now we prove the result regarding the distribution of prime ideals within a circular sector.

Proof of the Angular Prime Ideal Theorem 1.17: Define the functions $\overline{f}(\phi)$ and $\underline{f}(\phi)$ as in Lemma 2.32 with $\delta = e^{-c_{26}\sqrt{\log x}}$.

Then we get

$$\begin{aligned}
\Pi(x;\phi_1,\phi_2) &= \sum_{\substack{\mathfrak{N}(\mathfrak{p})\le x \\ \phi_1\le\theta_{\mathfrak{p}}\le\phi_2}} 1 \le \sum_{\mathfrak{N}(\mathfrak{p})\le x} \overline{f}(\theta_{\mathfrak{p}}) \\
&= \sum_{\mathfrak{N}(\mathfrak{p})\le x} \sum_{-\infty}^{\infty} \overline{a}_n \chi^{wn}(\mathfrak{p}) \\
&= \overline{a}_0 \pi(x) + \sum_{n\ne 0} \overline{a}_n \Bigg(\sum_{\mathfrak{N}(\mathfrak{p})\le x} \chi^{wn}(\mathfrak{p}) \Bigg).
\end{aligned}$$

Thus by Theorems 1.13 and 2.30,

$$\begin{aligned}
\Pi(x;\phi_1,\phi_2) \le{}& \frac{1}{2\pi}(\phi_2-\phi_1+\delta)\big(Li(x) + O(xe^{-c_{27}\sqrt{\log x}})\big) \\
&+ O\Bigg(\sum_{n\ne 1} |\overline{a}_n| x e^{-c_{21}\frac{\log x}{\log(1+|n|)+\sqrt{\log x}}} \log^3(1+|n|) \Bigg). \qquad (10)
\end{aligned}$$

Analogously we deduce

$$\begin{aligned}
\Pi(x;\phi_1,\phi_2) \ge{}& \sum_{\mathfrak{N}(\mathfrak{p})\le x} \underline{f}(\theta_{\mathfrak{p}}) - \sum_{\mathfrak{N}(\mathfrak{p})\le x} \sum_{-\infty}^{\infty} \underline{a}_n \chi^{wn}(\mathfrak{p}) \\
={}& \frac{1}{2\pi}(\phi_2-\phi_1-\delta)\big(Li(x) + O(xe^{-c_{27}\sqrt{\log x}})\big) \\
&+ O\Bigg(\sum_{n\ne 1} |\underline{a}_n| x e^{-c_{21}\frac{\log x}{\log(1+|n|)+\sqrt{\log x}}} \log^3(1+|n|) \Bigg). \qquad (11)
\end{aligned}$$

We examine the sum on the right side of (9). By the bounds on $|\overline{a}_n|$ in Lemma 2.32 we get, if we split the sum in two parts:

$$\sum_{n\neq 1} |\overline{a}_n| x e^{-c_{21}\frac{\log x}{\log(1+|n|)+\sqrt{\log x}}} \log^3(1+|n|)$$
$$\ll x \sum_{1\leq n\leq \delta^{-2}} \frac{\log^3(n)}{n} e^{-c_{21}\frac{\log x}{\log(1+\delta^{-2}+\sqrt{\log x}}} + x \sum_{n>\delta^{-2}} \frac{\log^3(n)}{\delta n^2}.$$

For the first part we note that $\log(1+\delta^{-2}) \ll \log \delta^{-2} \ll \sqrt{\log x}$, and thus

$$\begin{aligned}
\sum_{1\leq n\leq \delta^{-2}} \frac{\log^3(n)}{n} e^{-c_{21}\frac{\log x}{\log(1+\delta^{-2}+\sqrt{\log x}}} &\leq \log^3(\delta^{-2}) e^{-c_{28}\sqrt{\log x}} \sum_{1\leq n\leq \delta^{-2}} \frac{1}{n} \\
&\leq e^{-c_{28}\sqrt{\log x}} \log^4(\delta^{-2}) \\
&= e^{-c_{28}\sqrt{\log x}} (2c_{26}\sqrt{\log x})^4 \\
&\leq e^{c_{29}} \sqrt{\log x}.
\end{aligned}$$

For the second part, we have

$$\sum_{n>\delta^{-2}} \frac{\log^3(n)}{\delta n^2} \ll \frac{1}{\delta} \frac{\log^3(\delta^{-2})}{\delta^{-2}} = \delta \log^3(\delta^{-2})$$
$$= e^{-c_{26}\sqrt{\log x}} (2c_{26}\sqrt{\log x})^3 \leq e^{-c_{30}\sqrt{\log x}}.$$

Obviously, we have the exact same bounds for the corresponding sum in (11) containing $\overline{a}_n$. Thus (10) and (11) yield

$$\left| \Pi(x; \phi_1, \phi_2) - \frac{\phi_2 - \phi_1}{2\pi} Li(x) \right| \ll x e^{-c_{25}\sqrt{\log x}},$$

and we are done. ■

3. Applications of the angular prime ideal theorem

3.1. **Variations on a Theme.** In this section, we discuss a few variations on the angular prime number theorem. In particular, we develop an angular prime *number* theorem for an imaginary quadratic ring.

In order to arrive at such a result, we recall a result of Landau from 1918 concerning the equidistribution of prime ideals in ideal classes.

As a reminder about ideal classes in a number field K, we denote $I(K)$ as the group of fractional ideals of K and $P(K)$ as its subgroup of prinicipal fractional ideals. Then, we define the class group of K as

$$Cl(K) = I(K)/P(K).$$

The order of $Cl(K)$ is denoted as h, the so-called class number. With this notation, we now state Landau's equidistribution theorem.

Theorem 3.1. (Landau, [15]) Fix an algebraic number field K with ring of algebraic integers $\mathcal{O}$, and let $\mathcal{C} \in Cl(K)$. Let $\Pi(x;\mathcal{C})$ denote the number of prime ideals in a fixed class $\mathcal{C}$ of $\mathcal{O}$ with norm at most x. Then for all $x \geq 3$, we have

$$\Pi(x;\mathcal{C}) = \frac{1}{h}Li(x) + O(xe^{-c\sqrt{\log x}}),$$

where c is a positive constant depending only on K.

Using Landau's Theorem in lieu of the Prime Ideal Theorem applies to an imaginary quadratic number ring, we obtain the following angular prime ideal theorem.

Theorem 3.2. Fix $-\frac{\pi}{w} < \phi_1 < \phi_2 \leq \frac{\pi}{w}$. Let $\Pi(x;\phi_1,\phi_2,\mathcal{C})$ denote the number of prime ideals in a fixed class $\mathcal{C}$ in a sector $[\phi_1,\phi_2]$ of an imaginary quadratic number ring $\mathcal{O}$ with norm at most x. Then, we have

$$\Pi(x;\phi_1,\phi_2,\mathcal{C}) = \left(\frac{\phi_2-\phi_1}{2\pi h}\right) Li(x) + O(x\exp(-\frac{b}{\sqrt{n}}\sqrt{\log x})),$$

where $n = [K:\mathbb{Q}]$, and b is a positive constant independent of K.

Corollary 3.3 follows if we consider the trivial class P of principal prime ideals; this corresponds to prime *elements* in $\mathcal{O}$ (up to associates). We call this an "Angular Prime Number Theorem".

Corollary 3.3. (**Angular Prime Number Theorem**)

Fix $-\frac{\pi}{w} < \phi_1 < \phi_2 \leq \frac{\pi}{w}$. Let $\Pi(x;\phi_1,\phi_2,P)$ denote the number of prime elements in a sector $[\phi_1,\phi_2]$ of an imaginary quadratic number ring $\mathcal{O}$ with norm at most x. Then, we have

$$\Pi(x;\phi_1,\phi_2,P) = \left(\frac{\phi_2-\phi_1}{2\pi h}\right) Li(x) + O(x\exp(-\frac{b}{\sqrt{n}}\sqrt{\log x})),$$

where $n = [K:\mathbb{Q}]$, and b is a positive constant independent of K.

Remark: We can also state a version of this theorem where we remove the restriction of ϕ_1 and ϕ_2 to simply being angles in $[0,2\pi)$. Since there is a "w to 1" correspondence of elements in $\mathcal{O}$ to principal ideals in $\mathcal{O}$, one simply multiplies the principal term of $\Pi(x;\phi_1,\phi_2,P)$ by a factor of w:

$$\Pi(x;\phi_1,\phi_2,P) = \left(\frac{w(\phi_2-\phi_1)}{2\pi h}\right) Li(x) + O(x\exp(-\frac{b}{\sqrt{n}}\sqrt{\log x})).$$

3.2. **Quotients of primes in an imaginary quadratic number ring.** Although it is a standard fact from Real Analysis that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, it may still come as a surprise that the set of quotients of *prime* numbers is a dense subset of $\mathbb{R}$. More recently, Garcia proved that the set of quotients of *prime* Gaussian integers is a dense subset of $\mathbb{C}$ [7].

Inspired by this, Sittinger [22] extended Garcia's result by using the angular prime number theorem (Corollary 3.3) to prove the following:

Theorem 3.4. (Sittinger, [22]) The set of quotients of primes in an imaginary quadratic ring $\mathcal{O}$ is dense in the complex plane.

Proof: It suffices to show that any annular sector $\{z \in \mathbb{C} : \psi_1 < \arg z < \psi_2,\ 0 < r < |z| < R\}$ contains a quotient of prime numbers in $\mathcal{O}$. Moreover, since associates of primes are again primes, we can assume without loss of generality that $\psi_1, \psi_2 \in [0, \frac{2\pi}{w}]$, where w denotes the number of units in $\mathcal{O}$.

For any $\theta \in (0, 2\pi]$, it follows from the angular prime number theorem that

$$\lim_{x\to\infty}\left[\Pi\Big(\frac{x}{r^2}; 0, \theta, P\Big) - \Pi\Big(\frac{x}{R^2}; 0, \theta, P\Big)\right] = \infty.$$

This means that there exists $x_0 > 0$ such that

$$\Pi\Big(\frac{x}{r^2}; 0, \theta, P\Big) - \Pi\Big(\frac{x}{R^2}; 0, \theta, P\Big) \geq 2 \text{ for all } x \geq x_0.$$

Moreover, the angular prime number theorem implies that there are infinitely many prime numbers in $\mathcal{O}$ in the sector (ψ_1, ψ_2). Therefore, there exists a prime number π_1 in the sector (ψ_1, ψ_2) with sufficiently large magnitude ($N(\pi_1) > x_0$) such that

$$\Pi\Big(\frac{|\pi_1|^2}{r^2}; 0, \xi, P\Big) - \Pi\Big(\frac{|\pi_1|^2}{R^2}; 0, \xi, P\Big) \geq 2,$$

where $\xi = \min\{\psi_2 - \arg(\pi_1),\ \arg(\pi_1) - \psi_1\}$.

Next, the inequality in the last assertion implies that there exists a prime number π_2 satisfying $\dfrac{|\pi_1|}{R} < |\pi_2| < \dfrac{|\pi_1|}{r}$ and $0 < \arg(\pi_2) < \xi$.

From this, it now follows that $r < \left|\dfrac{\pi_1}{\pi_2}\right| < R$ and $\psi_1 < \arg\left(\dfrac{\pi_1}{\pi_2}\right) < \psi_2$. ■

3.3. **Primes of the form $x^2 + ny^2$ in a sector.** In this section, we show how to apply the angular prime number theorem to give us sharper (natural) density results for certain positive definite quadratic forms.

As an illustration of such a result, we first consider an extension of the problem concerning of primes of the form $x^2 + 5y^2$ (for some integers x and y).

Example 3.5. Suppose that we want to know the density of primes of the form $x^2 + 5y^2$ that lie in the sector in the xy-plane bounded by $\phi_1 = \frac{\pi}{3}$ and $\phi_2 = \frac{\pi}{2}$ in the first quadrant.

The key idea is to note that $x^2+5y^2 = N(x+y\sqrt{-5})$ in $\mathcal{O} = \mathbb{Z}[\sqrt{-5}]$. Then, it is equivalent to find the density of prime elements in $\mathcal{O}$ that have prime norm.

Since such elements are in 1-1 correspondence with generators of principal ideals in $\mathcal{O}$, we consider the principal ideal class in $\mathcal{O}$. As it is known in $\mathcal{O}$ that $h = 2$, Corollary 3.3 yields

$$\Pi(x; \frac{\pi}{3}, \frac{\pi}{2}, P) = \frac{1}{24} Li(x) + O\big(x \exp(-\frac{b}{\sqrt{2}}\sqrt{\log x})\big).$$

In other words, $\frac{1}{24}$ of the rational primes are of the form $x^2 + 5y^2$ and lie within the given sector.

We now state a theorem that generalizes what we have just seen in this last example. We content ourselves with considering the case that $d \not\equiv 1 \bmod 4$.

Theorem 3.6. Suppose that $d > 0$ and $d \not\equiv 1 \bmod 4$. Then, the density of rational primes of the form $x^2 + dy^2$ lying between lines passing through the origin with arguments $0 \leq \varphi_1 < \varphi_2 \leq \frac{\pi}{2}$ is equal to $\dfrac{\varphi_2 - \varphi_1}{2\pi h}$, where h is the class number of $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$.

Proof: Consider the imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-d})$. Since $d \not\equiv 1 \bmod 4$, we have $\mathcal{O}_K = \mathbb{Z}[\sqrt{-d}]$. Then, noting that $x^2 + dy^2 = N(x + y\sqrt{-d})$, it suffices to find the density of elements in $\mathcal{O}$ that have prime norm (and are thus primes in $\mathcal{O}$) and are in the sector (φ_1, φ_2). Since such elements (are in 1-1 correspondence with generators of principal prime ideals in $\mathcal{O}$, we consider the principal ideal class in $\mathcal{O}$. Then, Corollary 3.3 yields the desired result. ■

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