

Properties of \mathbb{A}^3 —Ellipses and Their Zariski—closures

By

Maria Bernadette Baroi-Chavez

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENT FOR THE DEGREE OF

MASTER'S IN SCIENCE
IN MATHEMATICS

at the

CALIFORNIA STATE UNIVERSITY - CHANNEL ISLANDS

2018


© 2018

Maria Bernadette Baroi-Chavez

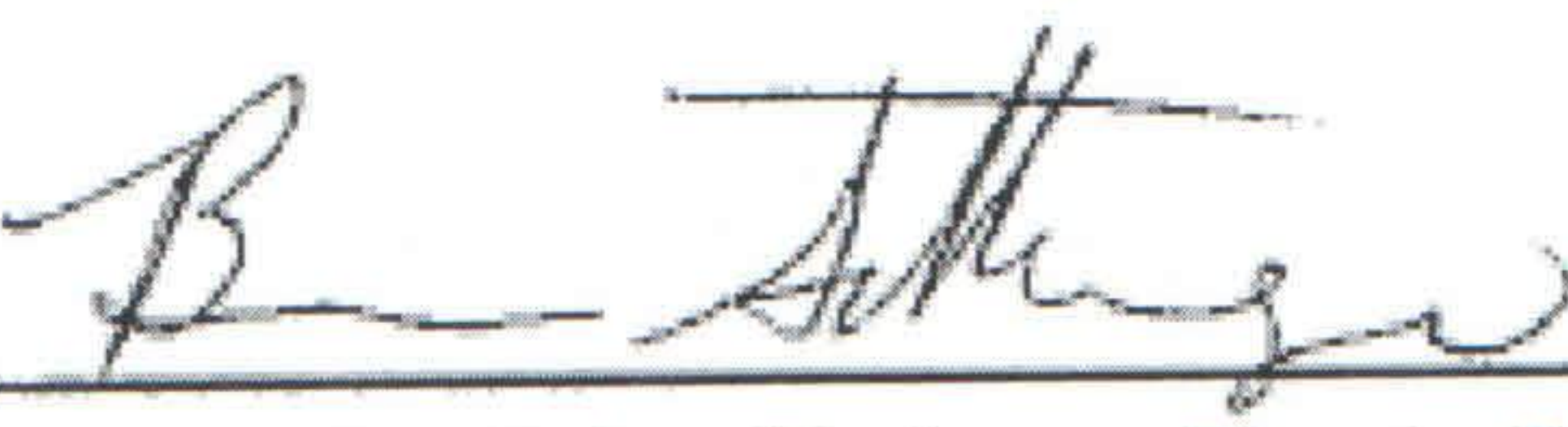
ALL RIGHTS RESERVED

Signature Page for the Master's in Mathematics Thesis
Maria Bernadette Baroi-Chavez

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

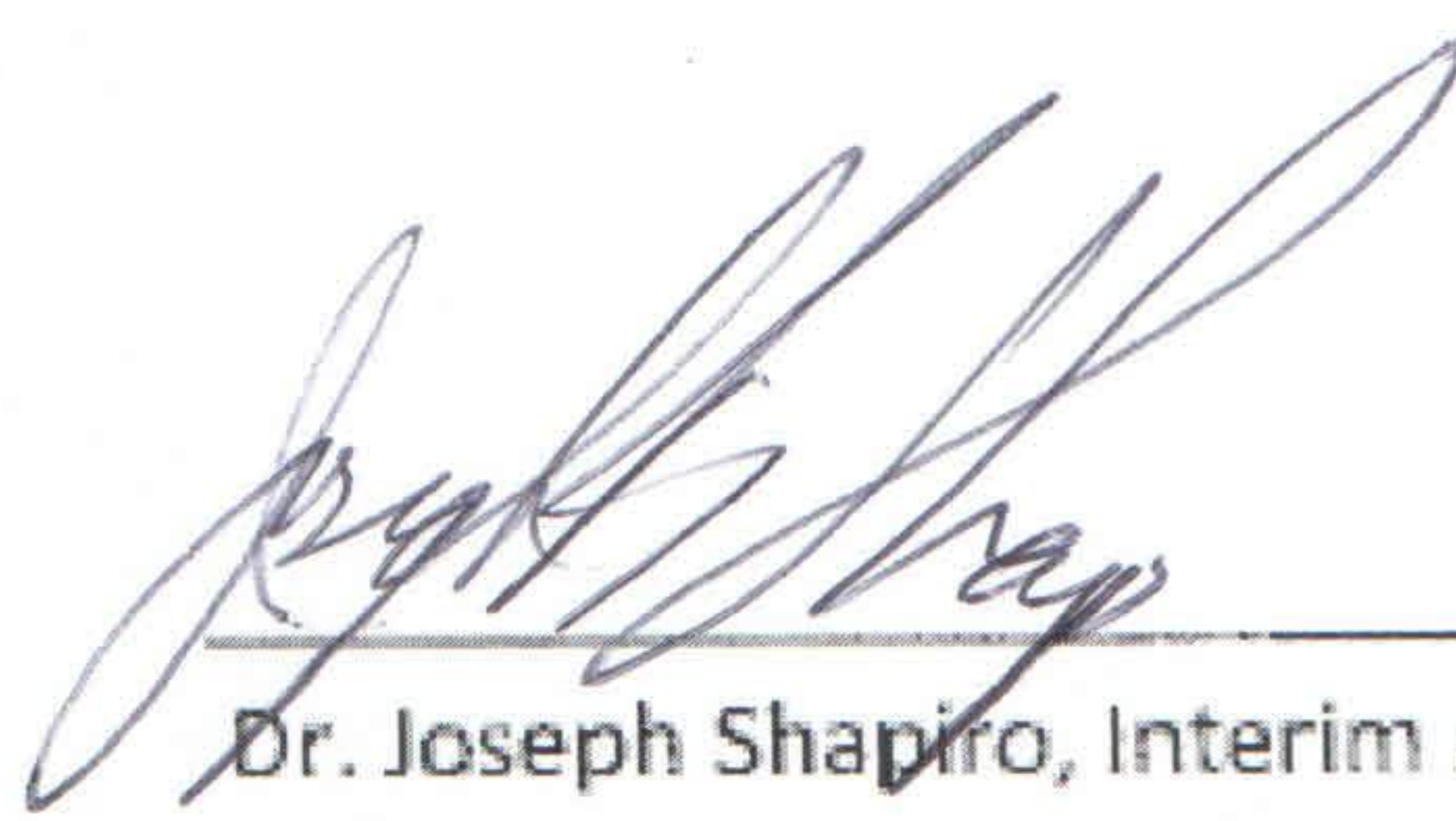


Dr. Ivona Grzegorczyk, Thesis Advisor
17 May 2018
Date



Dr. Brian Sittinger, Thesis Committee
10 May 2018
Date

APPROVED FOR THE UNIVERSITY



Dr. Joseph Shapiro, Interim AVP & Dean, Graduate Programs
May 24, 2018
Date

Non-Exclusive Distribution License

In order for California State University Channel Islands (CSUCI) to reproduce, translate and distribute your submission worldwide through the CSUCI Institutional Repository, your agreement to the following terms is necessary. The author(s) retain any copyright currently on the item as well as the ability to submit the item to publishers or other repositories.

By signing and submitting this license, you (the author(s) or copyright owner) grants to CSUCI the nonexclusive right to reproduce, translate (as defined below), and/or distribute your submission (including the abstract) worldwide in print and electronic format and in any medium, including but not limited to audio or video.

You agree that CSUCI may, without changing the content, translate the submission to any medium or format for the purpose of preservation.

You also agree that CSUCI may keep more than one copy of this submission for purposes of security, backup and preservation.

You represent that the submission is your original work, and that you have the right to grant the rights contained in this license. You also represent that your submission does not, to the best of your knowledge, infringe upon anyone's copyright. You also represent and warrant that the submission contains no libelous or other unlawful matter and makes no improper invasion of the privacy of any other person.

If the submission contains material for which you do not hold copyright, you represent that you have obtained the unrestricted permission of the copyright owner to grant CSUCI the rights required by this license, and that such third party owned material is clearly identified and acknowledged within the text or content of the submission. You take full responsibility to obtain permission to use any material that is not your own. This permission must be granted to you before you sign this form.

IF THE SUBMISSION IS BASED UPON WORK THAT HAS BEEN SPONSORED OR SUPPORTED BY AN AGENCY OR ORGANIZATION OTHER THAN CSUCI, YOU REPRESENT THAT YOU HAVE FULFILLED ANY RIGHT OF REVIEW OR OTHER OBLIGATIONS REQUIRED BY SUCH CONTRACT OR AGREEMENT.

The CSUCI Institutional Repository will clearly identify your name(s) as the author(s) or owner(s) of the submission, and will not make any alteration, other than as allowed by this license, to your submission.

Properties of β -Ellipses and Their Zariski-Closures

Title of Item

Algebraic Geometry, β -Ellipses, k -ellipses, Zariski-closures

3 to 5 keywords or phrases to describe the item

Maria Bernadette Baroi-Chavez

Author(s) Name (Print)

MB

Author(s) Signature

05/17/18

Date

This is a permitted, modified version of the Non-exclusive Distribution License from MIT Libraries and the University of Kansas.

DEDICATION

I would like to dedicate this work to the three most important men in my life. First and foremost, my son, Justice Sreshtho Chavez. His presence taught me how to be efficient with my time and stay focused on important tasks, so I can do even more important things like spend time cuddling with him. Secondly, my Baba, Mr. Baroi, whose example inspired me to pursue a career in Mathematics and helped me build my own dream around it. And finally, my husband, Rick, whose love and support never let me give up on that dream, even though there were many times I wanted to.

ACKNOWLEDGEMENTS

“Properties of 3-Ellipses and Their Zariski-closures” is the brainchild of my advisor, Dr. Ivona Grzegorczyk, Chair of the Mathematics Department at California State University Channel Islands.

This research is the continuation and aftermath of the projects done by each student enrolled in our Algebraic Geometry class during the spring of 2017. My goal was to create an overview of all the findings we had made as a class and build upon those findings. In that effort, I have extensively used the work of Vickie Chen, Lazarus Dixon, Osbelia Dueñas, and Bobby Martinez. With much appreciation, I would also like to mention Vincent Ferguson and David Lieberman; although I have not included their results in this specific paper, their works have deepened my understanding of algebraic geometry.

In addition, this research is also a continuation of Troy Jasso’s, John Stebbins’, and Kristin Tejeda’s graduate work. Their results laid the foundation for my work, and for this, I am grateful to them.

I would also like to express my gratitude toward the ladies of my Women’s Bible Study Group. Without their prayers and encouragement, I would have never made it this far.

Finally, I am thankful to my God for giving me the ability, patience, and persistence to do this work.

Camarillo, California

May 10, 2018

ABSTRACT

We study the Zariski-closures varieties of 3 –ellipses defined by foci that are either lines or points in $\mathbb{R}^3(x, y, z)$ as algebraic surfaces in three-dimensional real projective space, $\mathbb{P}^3(x, y, z, w)$. These surfaces are the smallest algebraic varieties containing the 3 –ellipses. In our case, they are described as the sets of zeroes of homogeneous polynomial functions of degree eight. Geometrically, these types of surfaces in \mathbb{P}^3 are yet to be classified. Using algebraic geometry and new visualization tools, such as Surfer and (Wolfram) Mathematica, we study the properties of these surfaces with different foci configurations. In particular, we study the shapes and symmetries of each surface, their reducibility, boundedness, etc., and compare the results with the already existing ones. We also study the stability of these properties under deformations by applying different deformation techniques on the surfaces and analyzing the results.

TABLE OF CONTENTS

1. Introduction	1
2. History and Results	3
3. Definitions	4
4. Methodology	7
Two intersecting lines and a point	
4.1 Two intersecting lines and the point of intersection	
4.2 Two intersecting lines and a point on one of the lines	
4.3 Two intersecting lines and a point on neither of the lines	
Two skew lines and a point	
4.4 Two skew lines and a point on one of the lines	
4.5 Two skew lines and a point on neither of the lines	
Two parallel lines and a point	
4.6 Two parallel lines and a point on one of the lines	
4.7 Two parallel lines and a point on neither of the lines	
5. Results and Observation	22
6. Comparison	37
7. Deformations	38
8. Conclusion	40
9. Bibliography	41

INTRODUCTION

Over the past centuries, numerous scholars have encountered k –ellipses in different contexts. In n –dimensional Euclidean geometry, the first generalizations of the traditional ellipse are **k –ellipses** with k point foci. More precisely, for a finite number of k points (called foci) in \mathbb{R}^n with coordinates, (u_{k1}, \dots, u_{kn}) , for each $n = 1, \dots, k$, a k –ellipse is the locus of all points whose sum of distances to the k foci is a constant d . If not empty, the k -ellipse in \mathbb{R}^n is simply the set:

$$\left\{ (x_1, \dots, x_n), \in \mathbb{R}^n \mid \sum_{j=1}^k \sqrt{\sum_{i=1}^n (x_i - u_{ji})^2} = d \right\}.$$

For example in \mathbb{R}^2 , a 1–ellipse is a circle and a 2–ellipse, or a standard ellipse, is the set of all points such that the sum of their distances from two fixed points (foci) are constant. For any number of foci k , the k –ellipse in \mathbb{R}^2 is a closed, convex curve [19]. The curve is smooth unless it goes through a focus [13]. Similarly when in \mathbb{R}^3 , a 1–ellipse is a sphere and a 2–ellipse is an ellipsoid, respectively.

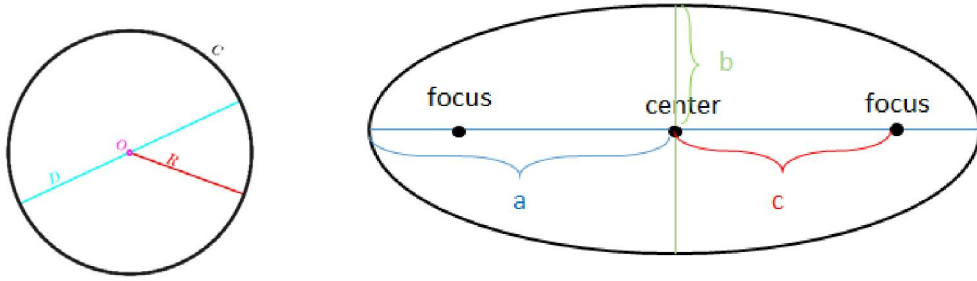


Figure 1: 1–ellipse and 2–ellipse in \mathbb{R}^2

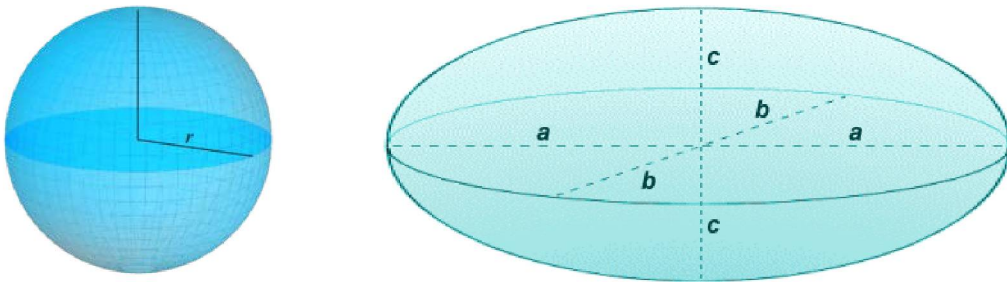


Figure 2: 1–ellipse and 2–ellipse in \mathbb{R}^3

In 1990, k –ellipses with foci consisting of points were found to be useful in solving optimization problems under constraints. About 15 years ago, flats in \mathbb{R}^n such as lines, planes, and hyperplanes, were introduced as foci for k –ellipses beside points, and the study of k –ellipses and related varieties in \mathbb{R}^n and in \mathbb{P}^n aided in solving convex optimization problems. Although, k –ellipses in \mathbb{R}^n with only points as foci have fully been described [13], the classification of generalized k –ellipses and the associated algebraic varieties still remains an unsolved problem. Keeping in mind the various possible configurations of foci, in this paper we explore 3 –ellipses having 3 –foci consisting of a combination of lines and points in \mathbb{R}^3 . In particular, we focus on the seven specific configurations listed below:

- two perpendicular intersecting lines and the focal point of intersection
- two perpendicular intersecting lines and a point on one of the lines
- two perpendicular intersecting lines and a point on neither of the lines
- two perpendicular skew lines and a point on one of the lines
- two perpendicular skew lines and a point on neither of the lines
- two parallel lines and a point on one the lines
- two parallel lines and a point on neither of the lines (also not co-planer with the lines)

We study the properties of the 3 –ellipses and their associated Zariski-closure varieties generated by the above configurations. Then we comment on our results in the wider algebraic geometry context.

HISTORY AND RESULTS

In this chapter we start with several of the known results of the k –ellipses with foci f_1, \dots, f_k (not necessarily points). Some of these results involve only points as foci and the others include some flats as well. We are using here works of Stebbins [16] and Tejeda [17], as well as the fundamental paper of Nie, Parrilo, and Sturmfel [13].

NOTE: Some of the terms in the statements below are defined more explicitly in Chapter 3.

Theorem 2. 1. The defining irreducible polynomial $P_k^n(\mathbf{x}) = P_k^n(x_1, x_2, \dots, x_n)$ of the k –ellipse with k focal points is monic of degree 2^k in the parameter d . It has degree 2^k in (x_1, x_2, \dots, x_n) if k is odd, and it has degree $2^k - \binom{k}{k/2}$ if k is even [13].

Theorem 2. 2. Let E be a generalized k –ellipse in \mathbb{R}^n with foci f_1, \dots, f_k (not necessarily points). If the set of foci has a non-trivial symmetry group G , then E also has the same symmetry group [17].

Theorem 2. 3. Let E be a generalized k -ellipse in \mathbb{R}^n with foci f_1, \dots, f_k . If at least one of the foci is a point, then the generalized k -ellipse in \mathbb{R}^n is bounded [17].

Theorem 2. 4. Let E be a generalized k -ellipse in \mathbb{R}^n with foci f_1, \dots, f_k . Then, E is non-singular if it does not pass through any of the foci [17].

Theorem 2. 5. Let E be a generalized k -ellipse in \mathbb{R}^n with foci f_1, \dots, f_k . Then one of the following holds:

- a) If there exists a hyperplane that is perpendicular to all the foci of the generalized k –ellipse, then the generalized k –ellipse is unbounded.
- b) If there does not exist a hyperplane that is perpendicular to all the foci of the generalized k –ellipse, then the generalized k -ellipse is bounded [17].

We use the above results in our work and discuss our finding in similar context.

DEFINITIONS

Definition 1. **Affine Space** of dimension n over a field k is defined as a set of points:

$$A^n := A_k^n := k^n = \{(a_1, a_2, \dots, a_n) : a_i \in k\}, \text{ where } (a_1, a_2, \dots, a_n) \text{ are points with coordinates.}$$

REMARK: We call the point $\mathbf{0} = (0, \dots, 0)$ the origin. We consider \mathbb{R}^n and \mathbb{C}^n as affine spaces.

Definition 2. **The k -ellipse** in \mathbb{R}^n (if not empty) is the hypersurface consisting of all points, the sum of whose distances from the k given foci, that are either points, lines, planes or other flats, is equal to some positive fixed real number d . k foci with coordinates, (u_{k1}, \dots, u_{kn}) , where $n = 1, \dots, k$, define

$$\text{a } k\text{-ellipse in } \mathbb{R}^n \text{ as the set } \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^k \sqrt{\sum_{i=1}^n (x_i - u_{ji})^2} = d \right\}.$$

Example 2.1 An ellipse in \mathbb{R}^2 is a collection of points such that the sum of the distances from the foci $d_1 + d_2$ is equal to a constant d . (See Figure 1)

Definition 3. **The Fermat-Weber distance**, denoted as d_w , is the least non-negative real number such that the generalized k -ellipse E is non-empty.

NOTE: For our purposes we will always assume $d \geq d_w$.

Definition 4. **An Algebraic Variety** $V = V(f_1, \dots, f_m)$ in \mathbb{R}^n (or \mathbb{C}^n) is the set of points satisfying a finite system of polynomial equations $f_i(x_1, \dots, x_n) = 0$, for $i = 1, 2, \dots, m$.

REMARK: In other words, a variety is the set of common zeros of several polynomials. In classical algebraic geometry, the polynomials may have complex numbers as coefficients. Note that such polynomials always have zeros. For example, $\{(x, y, z) : x^2 + y^2 - z^2 = 0\}$ is a cone V , and $\{(x, y, z) : x^2 + y^2 - z^2 = 0, ax + by + cz = 0\}$ is a conic section, which is a sub-variety of the cone V . When a variety is embedded in a projective space \mathbb{P}^3 with homogeneous coordinates x_0, x_1, x_2, x_3 it is called a **projective algebraic variety**.

Example 4.1 Consider the following system of equations in \mathbb{R}^2 :

$$\begin{cases} f_1(x, y) = xy = 0 \\ f_2(x, y) = x^2 + y^2 - 1 = 0 \end{cases}$$

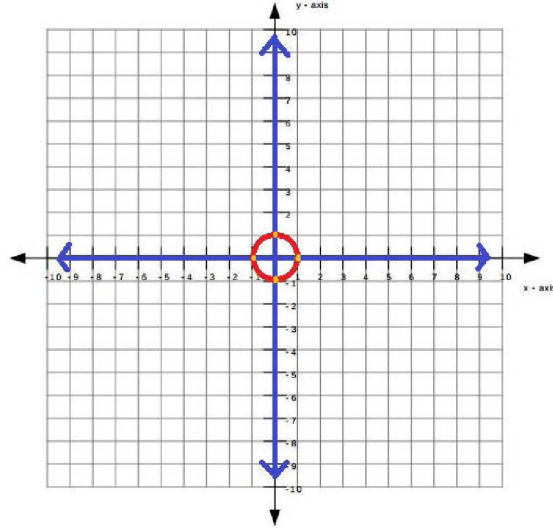


Figure 3: Example of an algebraic variety in \mathbb{R}^2

Here the $V(f_1)$ is the sum of the x and the y – axis and $V(f_2)$ is the unit circle. Therefore,

$$V(f_1, f_2) = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}.$$

Definition 5. Projective Space: Let k^n be a finite dimensional vector space over an arbitrary algebraically closed field k . The projective space $\mathbb{P}(k^{n+1})$ is the set of equivalence classes of all lines passing through the origin in k^{n+1} . We define $\mathbb{P}^n := (k^{n+1} - \{0\})/\sim$, where \sim is the equivalence relation $(x_0, x_1, \dots, x_n) = (\lambda x_0, \lambda x_1, \dots, \lambda x_n)$, where λ is an arbitrary non-zero real number.

NOTE: In our case $k = \mathbb{R}$ or \mathbb{C} .

Example 5.1 The following relation defines the three-dimensional real projective space with variables (x, y, z, w) , $\mathbb{P}^3(x, y, z, w) = (\mathbb{R}^4 - \{0\})/\sim$, where \sim represents the equivalence relation $(x, y, z, w) = (\lambda x, \lambda y, \lambda z, \lambda w)$ for any non-zero real number λ . More precisely, a three-dimensional real projective space is the set of classes of all lines in \mathbb{R}^4 passing through the origin, where each line is represented by exactly one point lying on it. Consider all the unit vectors attached at the origin, their end-points form a sphere S^3 in \mathbb{R}^4 and each of them represents a line passing through it. If we ‘glue’ the end-points of the vectors representing the same line in S^3 , we get $\mathbb{P}^3_{\mathbb{R}}(x, y, z, w)$ that contains the usual Euclidean $\mathbb{R}^3(x, y, z, 1)$ as a Zariski-open subset [7]. In other words, $\mathbb{P}^3_{\mathbb{R}}(x, y, z, w)$ is a compactification of \mathbb{R}^3 by a plane at infinity.

Definition 6. Zariski-Closure in A^n : Algebraic varieties in A^n are considered closed sets in Zariski topology. The complements of a Zariski-closed sets are Zariski-open sets. Such defined Zariski topology is well-defined on A^n . The Zariski closure of a subset $Z \subset A^n$ is the smallest algebraic variety set V of containing Z . V is the set of zeros of an ideal $I = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$ generated by polynomials vanishing (at least on Z), denoted $V(I(Z))$.

Definition 7. Irreducible/Reducible Algebraic Variety: An algebraic variety is called irreducible if it cannot be written as the union of distinct nonempty algebraic varieties. Otherwise, it is called reducible.

Example 7.1 The algebraic variety V defined by $xy = 0$ is reducible, because it is the union of the solutions of two varieties, V_1 and V_2 given by two linear polynomials $x = 0$ and $y = 0$, respectively.

Definition 8. Singularity: A singular point (singularity) of an algebraic surface V is a point P at which the unique tangent plane to the surface does not exist or is not well defined.

Example 8.1 In \mathbb{R}^3 , all singular points on a surface $V(f)$, defined by a polynomial f , are the solutions to the following the vector equation $\nabla f = (0, 0, 0)$.

Definition 9. The Dimension of a Variety: for an irreducible algebraic variety V , the dimension of V is the dimension of the tangent vector space at any non-singular point of V . This is the algebraic analogue to the fact that a connected manifold has a constant dimension.

Definition 10. Skew Lines: In Euclidean space, skew lines are two lines that do not intersect and are not parallel (see Figure 4).

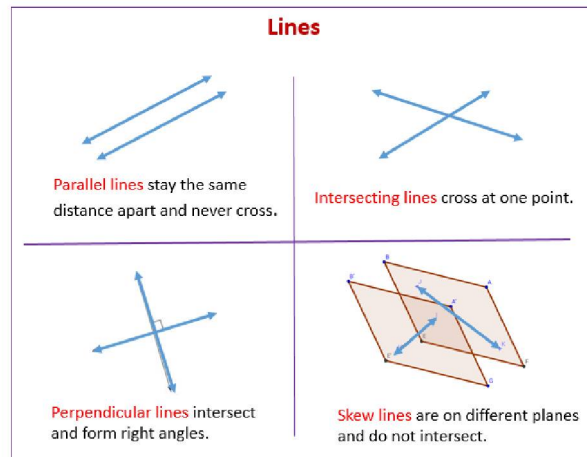


Figure 4: Possible configurations of lines in \mathbb{R}^3

METHODOLOGY

In this chapter, we present the equations for 3 –ellipses in \mathbb{R}^3 with various foci f_i . We start by considering the equation of a 3 –ellipse with point foci only. Let $f_1 = (x_1, y_1, z_1)$, $f_2 = (x_2, y_2, z_2)$ and $f_3 = (x_3, y_3, z_3)$ be the 3 focal points, and d be the distance greater than d_w (so the ellipse is not empty). Then, the equation of the 3-ellipse is the following:

$$E = \{(x, y, z) \in \mathbb{R}^3 : d_1((x, y, z), (x_1, y_1, z_1)) + d_2((x, y, z), (x_2, y_2, z_2)) + d_3((x, y, z), (x_3, y_3, z_3)) = d\}.$$

In other words, the equation of the 3-ellipse is

$$\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2} + \sqrt{(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2} = d.$$

Note that this equation includes radical expressions, hence does not define an algebraic variety, i.e. its set of solutions is just a subset E in \mathbb{R}^3 . The examples below show a construction of an algebraic variety containing E .

Example 1. Consider the circle (1 –ellipse) in \mathbb{R}^2 with focus $(0, 0)$ and radius d . By definition, the equation of E is $\sqrt{x^2 + y^2} = d$, which is not algebraic. Squaring both sides yields $x^2 + y^2 = d^2$, which is already an algebraic equation. Note that solutions sets for both equations are identical, this implies that circle is its own Zariski-closure.

Example 2. Now consider a 2 –ellipse in \mathbb{R}^2 with foci $(-1, 0)$ and $(1, 0)$, and distance $d = 4$. By definition, the equation of this 2 –ellipse is $\sqrt{(x + 1)^2 + y^2} + \sqrt{(x - 1)^2 + y^2} - 4 = 0$. Again, this equation is not algebraic. In order to make it algebraic, we need to remove the radicals. We multiply the expression on the left side by its three other conjugates as following:

$$\begin{aligned} & \left(\sqrt{(x + 1)^2 + y^2} + \sqrt{(x - 1)^2 + y^2} - 4 \right) \left(\sqrt{(x + 1)^2 + y^2} - \sqrt{(x - 1)^2 + y^2} - 4 \right) \left(-\sqrt{(x + 1)^2 + y^2} \right. \\ & \quad \left. + \sqrt{(x - 1)^2 + y^2} - 4 \right) \left(-\sqrt{(x + 1)^2 + y^2} - \sqrt{(x - 1)^2 + y^2} - 4 \right) = 0. \end{aligned}$$

Here it is important to note that because our original equation had 2 different radicals and each radical can assume two different signs, we had to multiply by a total of $2^2 = 4$ conjugates to make the equation completely radical-free.

Using Wolfram Alpha, we perform the multiplication and obtain the following algebraic equation:

$$(-48x^2 - 64y^2 + 192)^2 = 0.$$

Note that this equation is factorable and has double vanishing set. Hence, we take only one factor to obtain an irreducible variety and observe that 2-ellipse is also its own Zariski-closure defined by:

$$-48x^2 - 64y^2 + 192 = 0, \text{ or equivalently, } \frac{x^2}{(2)^2} + \frac{y^2}{(\sqrt{3})^2} = 1.$$

In general case of k focal points, we apply this technique to find algebraic varieties of associated to the the k -ellipses defined by the equations in \mathbb{R}^3 given at the beginning of this chapter:

$$\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} + \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2} + \sqrt{(x-x_3)^2 + (y-y_3)^2 + (z-z_3)^2} - d = 0.$$

Since this equation has 3 different radicals and each radical can assume two different signs, we multiply a total by $2^3 = 8$ conjugates and expect the equation resulting equation to be completely radical-free.

We denote the first, second and the third square root expression as a, b and c respectively, and obtain the following equation:

$$(A+B+C-d)(A-B+C-d)(A+B-C-d)(-A+B+C-d)(A-B-C-d) \\ (-A-B+C-d)(-A+B-C-d)(-A-B-C-d) = 0.$$

In essence, we multiply the radical expression on the left-hand side by its 7 different conjugates. The resulting expression has all variables in even degrees only, hence after the substitution of original variables, the expression is a polynomial. Therefore, it defines an algebraic variety V containing our 3-ellipse E . Sometimes this expression is factorable. When it is, we take the lowest degree factor(s) that vanishes on our 3-ellipse and call it the algebraic variety V defined by this factor the Zariski-closure variety of E in $\mathbb{R}^3(x, y, z)$. Otherwise, we take the entire non-factorable expression which defines an irreducible algebraic variety as Zariski-closure for the 3-ellipse.

Now consider a 3-ellipse in \mathbb{R}^3 with one or more of the foci that are lines. Without loss of generality, we can choose f_1 to be a line, and we can assume it is the x -axis (as we can translate and rotate the 3-ellipse without changing its properties. Then the distance equation between any point and this lines simplifies to $d_1((x, y, z), f_1) = \sqrt{y^2 + z^2}$. Now suppose that f_2 is also a line passing through two points $x_1 = (a_0, b_0, c_0)$ and $x_2 = (a + a_0, b + b_0, c + c_0)$ lying on it. To find the associated distance formula, we use the following parametric vector equation of the line:

$$\mathbf{v} = \begin{pmatrix} at + a_0 \\ bt + b_0 \\ ct + c_0 \end{pmatrix}.$$

Then, the squared distance between a point on the line with parameter t and a point $\mathbf{x} = (x, y, z)$ is therefore

$$D = d^2 = [(a_0 - x) + at]^2 + [(b_0 - y) + bt]^2 + [(c_0 - z) + ct]^2.$$

To minimize the distance, we set $\frac{d(D)}{dt} = 0$ and solve for t to obtain $t = -\frac{(\mathbf{x}_1 - \mathbf{x}) \cdot (\mathbf{x}_2 - \mathbf{x}_1)}{|\mathbf{x}_2 - \mathbf{x}_1|^2}$, where the symbol "." denotes the vector dot product. Then by substituting t back into the previous equation we find the minimum distance as

$$\begin{aligned} d^2 &= (a_0 - x)^2 + (b_0 - y)^2 + (c_0 - z)^2 + 2t[a(a_0 - x) + b(b_0 - y) + c(c_0 - z)] + t^2[a^2 + b^2 + c^2] \\ &= |\mathbf{x}_1 - \mathbf{x}|^2 - 2 \frac{[(\mathbf{x}_1 - \mathbf{x}) \cdot (\mathbf{x}_2 - \mathbf{x}_1)]^2}{|\mathbf{x}_2 - \mathbf{x}_1|^2} + \frac{[(\mathbf{x}_1 - \mathbf{x}) \cdot (\mathbf{x}_2 - \mathbf{x}_1)]^2}{|\mathbf{x}_2 - \mathbf{x}_1|^2} \\ &= \frac{|\mathbf{x}_1 - \mathbf{x}|^2 |\mathbf{x}_2 - \mathbf{x}_1|^2 + [(\mathbf{x}_1 - \mathbf{x}) \cdot (\mathbf{x}_2 - \mathbf{x}_1)]^2}{|\mathbf{x}_2 - \mathbf{x}_1|^2}. \end{aligned}$$

Since $(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2 \mathbf{B}^2 - (\mathbf{A} \cdot \mathbf{B})^2$, where \times denotes the vector cross product, we have

$$d^2 = \frac{|(\mathbf{x}_2 - \mathbf{x}_1) \times (\mathbf{x}_1 - \mathbf{x})|^2}{|\mathbf{x}_2 - \mathbf{x}_1|^2}.$$

Taking square root of both sides we obtain,

$$d = \frac{|(\mathbf{x}_2 - \mathbf{x}_1) \times (\mathbf{x}_1 - \mathbf{x})|}{|\mathbf{x}_2 - \mathbf{x}_1|} = \frac{|(\mathbf{x} - \mathbf{x}_1) \times (\mathbf{x} - \mathbf{x}_2)|}{|\mathbf{x}_2 - \mathbf{x}_1|}.$$

Therefore, the distance between a point $\mathbf{x} = (x, y, z)$ and the line $(at + a_0, bt + b_0, ct + c_0)$ defined by the points $\mathbf{x}_1 = (a_0, b_0, c_0)$ and $\mathbf{x}_2 = (a + a_0, b + b_0, c + c_0)$ can be written as

$$\begin{aligned} d &= \frac{|(a, b, c) \times (a_0 - x, b_0 - y, c_0 - z)|}{|(a, b, c)|} \\ d &= \frac{|b(c_0 - z) - c(b_0 - y), c(a_0 - x) - a(c_0 - z), a(b_0 - y) - b(a_0 - x)|}{|(a, b, c)|} \\ d &= \frac{\sqrt{(b(c_0 - z) - c(b_0 - y))^2 + (c(a_0 - x) - a(c_0 - z))^2 + (a(b_0 - y) - b(a_0 - x))^2}}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

We can write it as

$$d = \frac{\sqrt{M}}{\sqrt{a^2 + b^2 + c^2}},$$

where

$$\begin{aligned} M &= a^2 c_0^2 + b^2 c_0^2 - 2bcc_0 b_0 + a^2 b_0^2 + c^2 b_0^2 - 2acc_0 a_0 - 2abb_0 a_0 + b^2 a_0^2 + c^2 a_0^2 + 2acc_0 x \\ &\quad + 2abb_0 x - 2b^2 a_0 x - 2c^2 a_0 x + b^2 x^2 + c^2 x^2 + 2bcc_0 y - 2a^2 b_0 y - 2c^2 b_0 y \\ &\quad + 2aba_0 y - 2abxy + a^2 y^2 + c^2 y^2 - 2a^2 c_0 z - 2b^2 c_0 z + 2bcb_0 z + 2aca_0 z - 2acxz \\ &\quad - 2bcyz + a^2 z^2 + b^2 z^2. \end{aligned}$$

We calculate the distance between a point $x = (x, y, z)$ and the y –axis. By choosing two random points $(0, -1, 0)$ and $(0, 1, 0)$ on the axis, we have $(a_0, b_0, c_0) = (0, -1, 0)$, $(a + a_0, b + b_0, c + c_0) = (0, 1, 0)$ and then solve for a, b and c . Substituting $a_0 = 0, b_0 = -1, c_0 = 0, a = 0, b = 2, c = 0$ in the above equation we obtain $d = \frac{\sqrt{4x^2 + 4z^2}}{2} = \sqrt{\frac{4x^2 + 4z^2}{4}} = \sqrt{x^2 + z^2}$.

Therefore, the distance between a point $x = (x, y, z)$ and the y –axis is given by $\sqrt{x^2 + z^2}$. Similarly, the distance between a point $x = (x, y, z)$ and the z –axis is $\sqrt{x^2 + y^2}$.

We use the above formula for our seven chosen configurations of foci to find Zariski closure algebraic varieties for the associated 3–ellipses.

3–ellipses with two perpendicular intersecting lines and a point as foci give as the following three cases to consider.

4.1 Two perpendicular intersecting lines and the point of intersection.

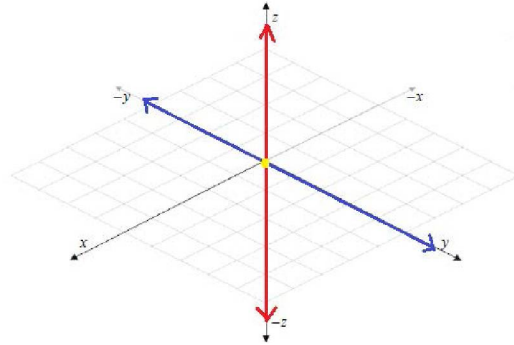


Figure 5: The 3–foci consisting of two perpendicular intersecting lines and the point of intersection

Consider the lines $L_1 = z$ –axis, $L_2 = y$ –axis in $\mathbb{R}^3(x, y, z)$ and take their point of intersection $P = (0, 0, 0)$ as foci. Then the 3–ellipse with $d = 2$ is given by

$$E_3 = \{(x, y, z) \in \mathbb{R}^3 : d_1((x, y, z), z\text{-axis}) + d_2((x, y, z), y\text{-axis}) + d_3((x, y, z), (0, 0, 0)) = 2\}.$$

Therefore, we have the equation $\sqrt{x^2 + y^2} + \sqrt{x^2 + z^2} + \sqrt{x^2 + y^2 + z^2} = 2$.

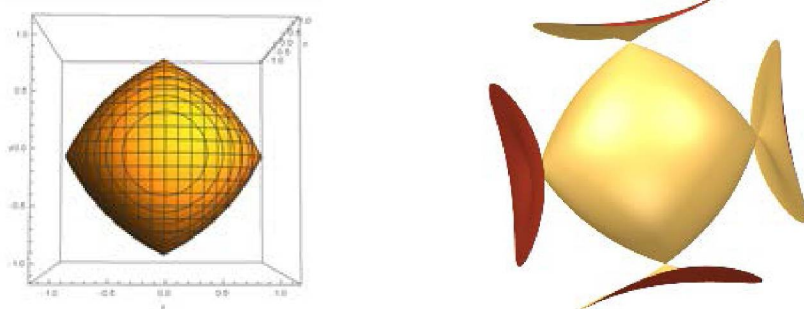


Figure 6: The picture to the left is the view of 3 –ellipse defined by two perpendicular intersecting lines and the focal point located at the intersection of the lines. The picture on the right shows a small part of the Zariski-closure variety as well.

To find the algebraic variety V in \mathbb{R}^3 , we multiply by conjugate expressions using (Wolfram) Mathematica. Hence, the equation of the Zariski-closure is

$$\begin{aligned}
 f(x, y, z) = & 256 - 768x^2 + 480x^4 - 112x^6 + 9x^8 - 512y^2 + 640x^2y^2 - 224x^4y^2 + 24x^6y^2 \\
 & + 256y^4 - 128x^2y^4 + 16x^4y^4 - 512z^2 + 640x^2z^2 - 224x^4z^2 + 24x^6z^2 + 384y^2z^2 \\
 & - 320x^2y^2z^2 + 56x^4y^2z^2 - 128y^4z^2 + 32x^2y^4z^2 + 256z^4 - 128x^2z^4 + 16x^4z^4 \\
 & - 128y^2z^4 + 32x^2y^2z^4 + 16y^4z^4.
 \end{aligned}$$

It is a degree 8 polynomial with 27 different terms.

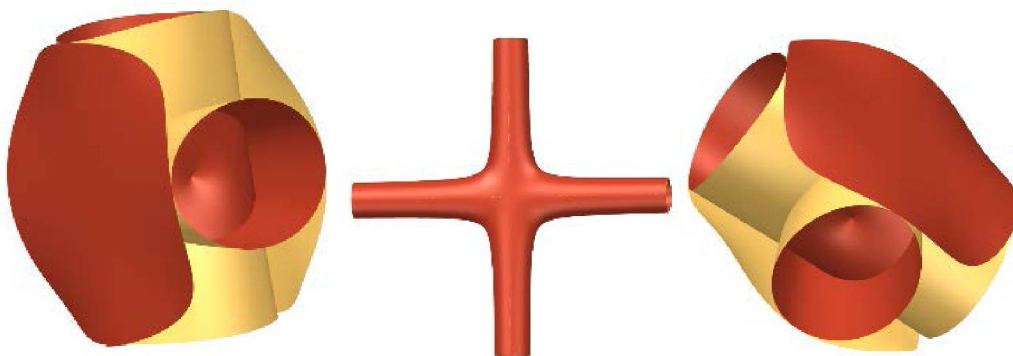


Figure 7: Various views of the Zariski-closure variety of the 3 –ellipse defined by two perpendicular intersecting lines and the focal point on the intersection of the lines

In this particular case we have the following symmetries of the set of foci, of the 3 –ellipse, and Zariski-closure variety generating the group: the mirror reflection in the yz – plane, the mirror reflection in the xz – plane, and the mirror reflection in the xy –plane, two 180° rotations about both x, y –axes, as well as one 90° rotation around z –axis.

4.2 Two perpendicular intersecting lines and a point on one of the lines (and not on the other):

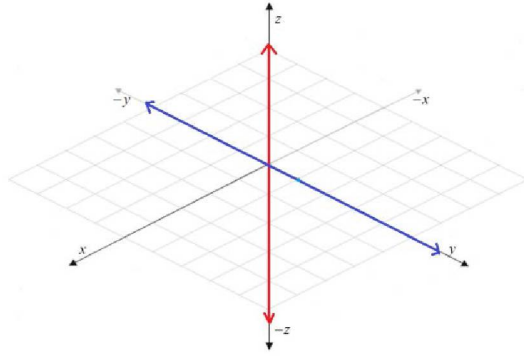


Figure 8: The 3 –foci consisting of two perpendicular intersecting lines and a point on one of the lines

Consider the lines $L_1 = z$ –axis, $L_2 = y$ –axis and the point $P = (0, 1, 0)$, which is on the line L_2 . To find the algebraic variety V in \mathbb{R}^3 , we choose the $d = 2$ and define the 3 – *ellipse* as following:

$$E_3 = \{(x, y, z) \in \mathbb{R}^3 : d_1((x, y, z), z\text{-axis}) + d_2((x, y, z), y\text{-axis}) + d_3((x, y, z), (0, 1, 0)) = 2\}.$$

Therefore, we obtain the equation $\sqrt{x^2 + y^2} + \sqrt{x^2 + z^2} + \sqrt{x^2 + (y - 1)^2 + z^2} = 2$.

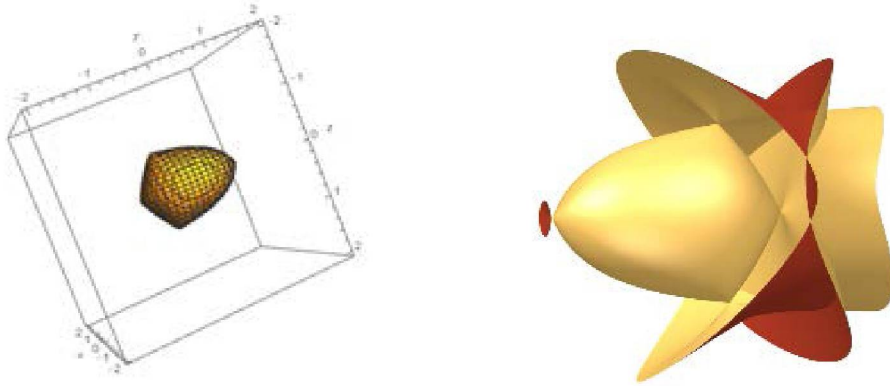


Figure 9: The picture on the left is the view of the 3 –ellipse defined by two perpendicular intersecting lines and a point on one of the lines. The second view shows part of the Zariski-closure as well.

Hence, the equation of the Zariski-closure is

$$\begin{aligned} f(x, y, z) = & 81 - 468x^2 + 366x^4 - 100x^6 + 9x^8 + 216y - 552x^2y + 232x^4y - 24x^6y - 72y^2 \\ & + 392x^2y^2 - 216x^4y^2 + 24x^6y^2 - 288y^3 + 320x^2y^3 - 32x^4y^3 + 144y^4 \\ & - 160x^2y^4 + 16x^4y^4 - 288z^2 + 504x^2z^2 - 208x^4z^2 + 24x^6z^2 - 384yz^2 \\ & + 288x^2yz^2 - 32x^4yz^2 + 56y^2z^2 - 336x^2y^2z^2 + 56x^4y^2z^2 + 416y^3z^2 \\ & - 32x^2y^3z^2 - 160y^4z^2 + 32x^2y^4z^2 + 256z^4 - 128x^2z^4 + 16x^4z^4 - 128y^2z^4 \\ & + 32x^2y^2z^4 + 16y^4z^4. \end{aligned}$$

Again, we obtain an 8th degree polynomial. The number of terms in this case is 39.

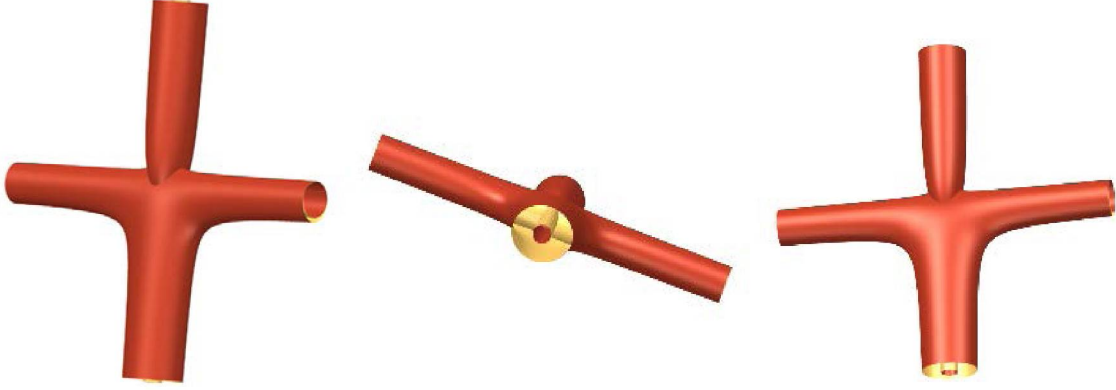


Figure 10: Various views of the Zariski-closure of the 3 –ellipse defined by two perpendicular intersecting lines and a point on one of the lines.

The symmetry group of the set of foci, of the ellipse, and of the Zariski-closure variety includes: the mirror reflection in the yz –plane: the mirror reflection in the xy –plane, one 180° rotation about the y –axis.

4.3 Two perpendicular intersecting lines and a point on neither of the lines:

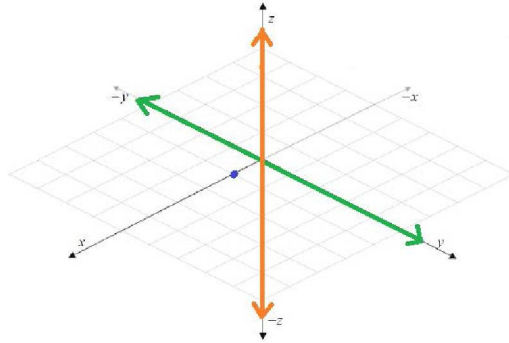


Figure 11: The 3 –foci consisting of two intersecting lines and a point that is on neither of the lines.

Consider the lines $L_1 = z$ –axis, $L_2 = y$ –axis and the point $P = (1, 0, 0)$, which is on neither of the lines.

To find the algebraic variety V in \mathbb{R}^3 , we choose the $d = 2$ and define the 3 – *ellipse* as following:

$$E_3 = \{(x, y, z) \in \mathbb{R}^3 : d_1((x, y, z), z\text{ – axis}) + d_2((x, y, z), y\text{ – axis}) + d_3((x, y, z), (1, 0, 0)) = 2\}.$$

Therefore, we obtain the equation $\sqrt{x^2 + y^2} + \sqrt{x^2 + z^2} + \sqrt{(x - 1)^2 + y^2 + z^2} = 2$.

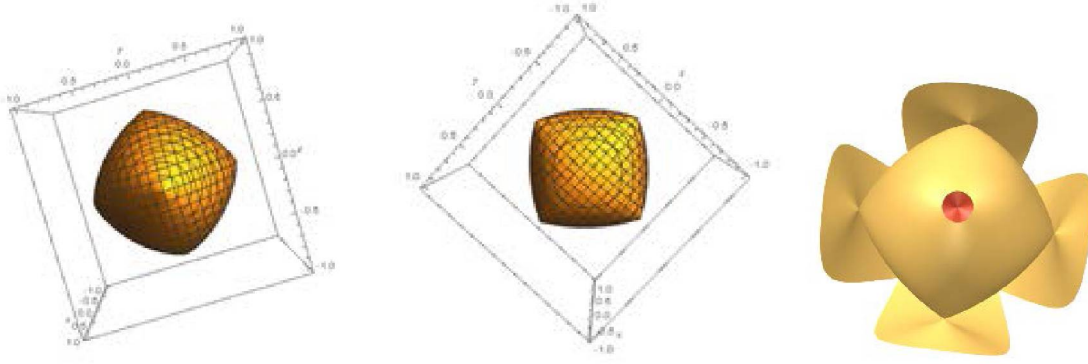


Figure 12: Two views of the 3 –ellipse defined by two perpendicular intersecting lines and a point on neither of the lines. The picture on the right shows part of the Zariski-closure as well.

Hence, the equation of the Zariski-closure is (Wolfram Mathematica)

$$\begin{aligned}
 f(x, y, z) = & 81 + 216x - 252x^2 - 456x^3 + 270x^4 + 264x^5 - 108x^6 - 24x^7 + 9x^8 - 288y^2 \\
 & - 384xy^2 + 376x^2y^2 + 288x^3y^2 - 240x^4y^2 - 32x^5y^2 + 24x^6y^2 + 256y^4 \\
 & - 128x^2y^4 + 16x^4y^4 - 288z^2 - 384xz^2 + 376x^2z^2 + 288x^3z^2 - 240x^4z^2 \\
 & - 32x^5z^2 + 24x^6z^2 + 184y^2z^2 + 416xy^2z^2 - 336x^2y^2z^2 - 32x^3y^2z^2 + 56x^4y^2z^2 \\
 & - 128y^4z^2 + 32x^2y^4z^2 + 256z^4 - 128x^2z^4 + 16x^4z^4 - 128y^2z^4 + 32x^2y^2z^4 \\
 & + 16y^4z^4.
 \end{aligned}$$

The degree of the polynomial above is 8 and the number of terms is 39.

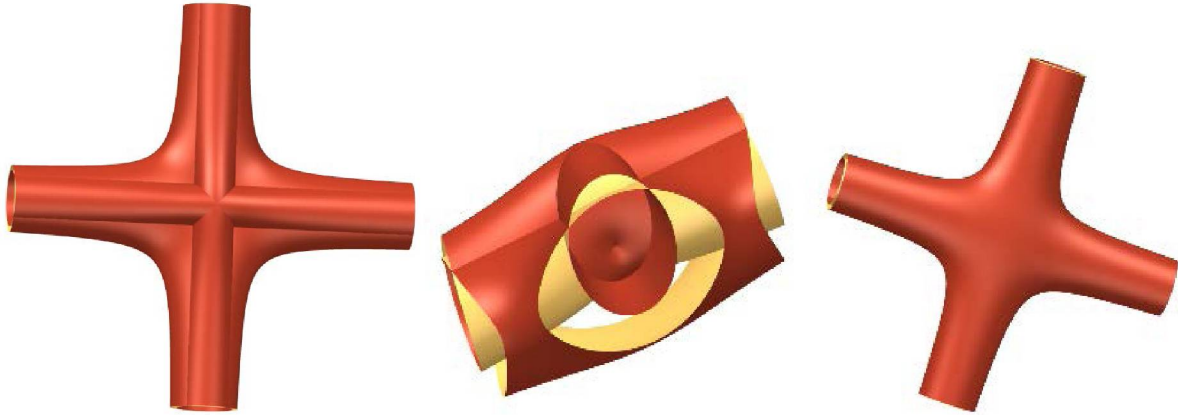


Figure 13: Various views of the Zariski-closure variety of the 3 –ellipse defined by two intersecting lines and a point on neither of the lines.

The symmetry group of the foci, the ellipse, and the Zariski-closure variety is generated by the mirror reflection in the xz – plane, the mirror reflection in the xy –plane, the 90° rotation about the x –axis.

3 –ellipses with two perpendicular skew lines and a point as foci gives as the two generic cases.

4.4 Two perpendicular skew lines and a point on one of the lines:

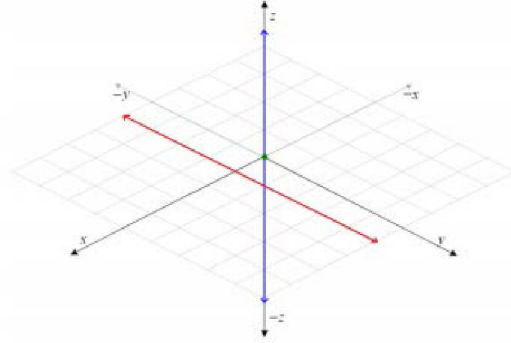


Figure 14: The 3-foci consisting of two perpendicular skew lines and a point on one of the lines.

Consider the skew lines $L_1 = z$ –axis, $L_2 =$ line parallel to y –axis= $(1, y, 0)$ and the point $P = (0, 0, 0)$, which is on the line L_1 . To find the algebraic variety V in \mathbb{R}^3 , we choose the $d = 4$ and define the 3 –*ellipse* as follows:

$$E_3 = \{(x, y, z) \in \mathbb{R}^3 : d_1((x, y, z), z\text{-axis}) + d_2((x, y, z), (1, y, 0)) + d_3((x, y, z), (0, 0, 0)) = 4\}.$$

Therefore, we obtain the equation $\sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + z^2} + \sqrt{x^2 + y^2 + z^2} = 4$.

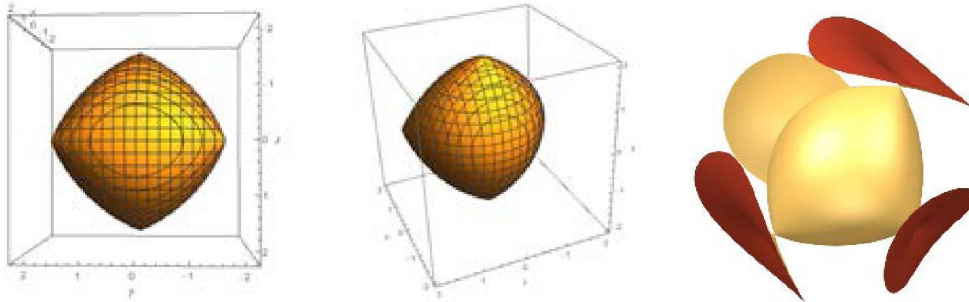


Figure 15: Two pictures on the left of the 3 –ellipse defined by two perpendicular skew lines and a point on one of the lines. The last view shows part of the Zariski-closure as well.

Hence, the equation of the Zariski-closure is (Wolfram Mathematica)

$$\begin{aligned} f(x, y, z) = & 50625 + 27000x - 38700x^2 - 9480x^3 + 6942x^4 + 936x^5 - 444x^6 - 24x^7 + 9x^8 \\ & - 30600y^2 - 4560xy^2 + 9896x^2y^2 + 1568x^3y^2 - 824x^4y^2 - 80x^5y^2 + 24x^6y^2 \\ & + 3600y^4 + 960xy^4 - 416x^2y^4 - 64x^3y^4 + 16x^4y^4 - 28800z^2 - 7680xz^2 \\ & + 9208x^2z^2 + 1056x^3z^2 - 912x^4z^2 - 32x^5z^2 + 24x^6z^2 + 5880y^2z^2 + 544xy^2z^2 \\ & - 1264x^2y^2z^2 - 96x^3y^2z^2 + 56x^4y^2z^2 - 480y^4z^2 - 64xy^4z^2 + 32x^2y^4z^2 \\ & + 4096z^4 - 512x^2z^4 + 16x^4z^4 - 512y^2z^4 + 32x^2y^2z^4 + 16y^4z^4. \end{aligned}$$

The degree of the above polynomial is 8 and the number of terms is 42.

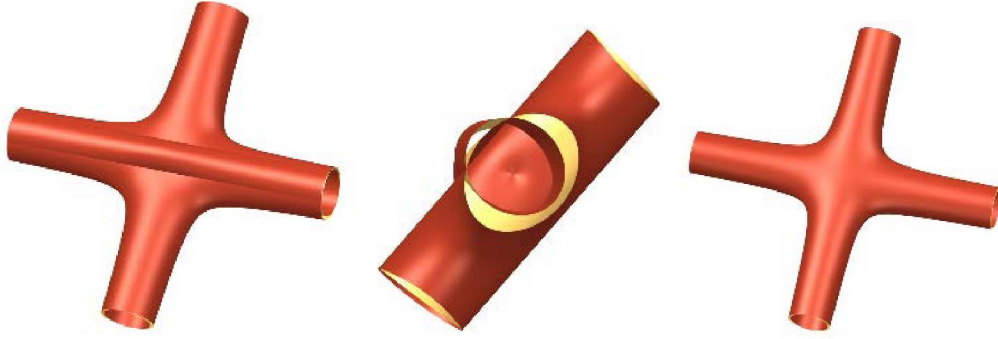


Figure 16: Various views of the Zariski-closure of the 3 –ellipse defined by two perpendicular skew lines and a point on one of the lines.

The symmetry group of the foci, the ellipse, and the Zariski-closure variety is generated by the mirror reflection in the xz – plane, and the mirror on the xy –plane, the 180° rotation about the x –axis.

4.5 Two perpendicular skew lines and a point on neither of the lines:

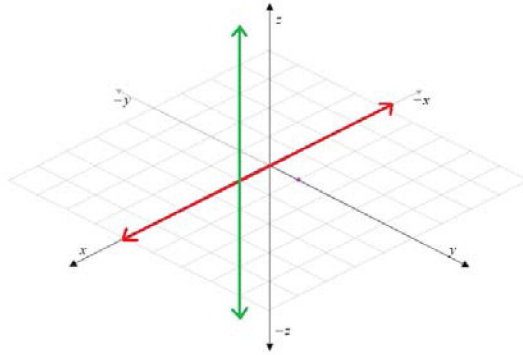


Figure 17: The 3-foci consisting of two skew lines and a point on neither of the lines.

Consider the two skew lines $L_1 =$ line parallel to the z –axis $= (0, -1, z)$, $L_2 = x$ –axis and the point $P = (0, 1, 0)$, which is not on either of the lines. To find the algebraic variety V in \mathbb{R}^3 , we choose the $d = 4$ and define the 3 –ellipse as following:

$$E_3 = \{(x, y, z) \in \mathbb{R}^3 : d_1((x, y, z), (0, -1, z)) + d_2((x, y, z), x\text{ – axis}) + d_3((x, y, z), (0, 1, 0)) = 4\}.$$

Therefore, we obtain the equation $\sqrt{x^2 + (y + 1)^2} + \sqrt{y^2 + z^2} + \sqrt{x^2 + (y - 1)^2 + z^2} = 4$.

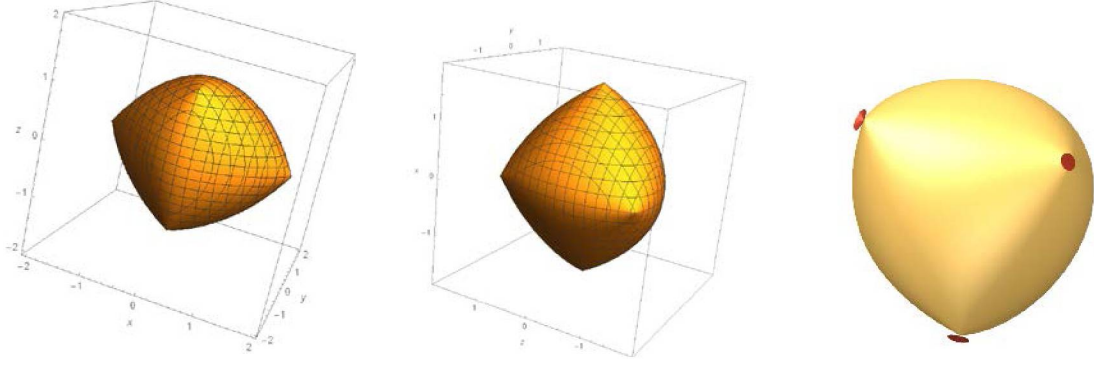


Figure 18: Two views of the 3 –ellipse defined by two perpendicular skew lines and a point on neither of the lines. The last view shows small part of the Zariski-closure as well.

The polynomial for the Zariski-closure variety becomes (Wolfram Mathematica)

$$\begin{aligned}
 f(x, y, z) = & 36864 - 24576x^2 + 4096x^4 - 33280y^2 + 7168x^2y^2 - 512x^4y^2 + 7952y^4 - 992x^2y^4 \\
 & + 16x^4y^4 - 520y^6 + 24x^2y^6 + 9y^8 - 27136z^2 + 5120x^2z^2 - 512x^4z^2 - 3072yz^2 \\
 & + 1024x^2yz^2 + 10912y^2z^2 - 1344x^2y^2z^2 + 32x^4y^2z^2 - 704y^3z^2 + 64x^2y^3z^2 \\
 & - 968y^4z^2 + 56x^2y^4z^2 + 48y^5z^2 + 24y^6z^2 + 3600z^4 - 480x^2z^4 + 16x^4z^4 \\
 & - 960yz^4 + 64x^2yz^4 - 416y^2z^4 + 32x^2y^2z^4 + 64y^3z^4 + 16y^4z^4.
 \end{aligned}$$

The degree of the polynomial is 8 and the number of terms is 35.

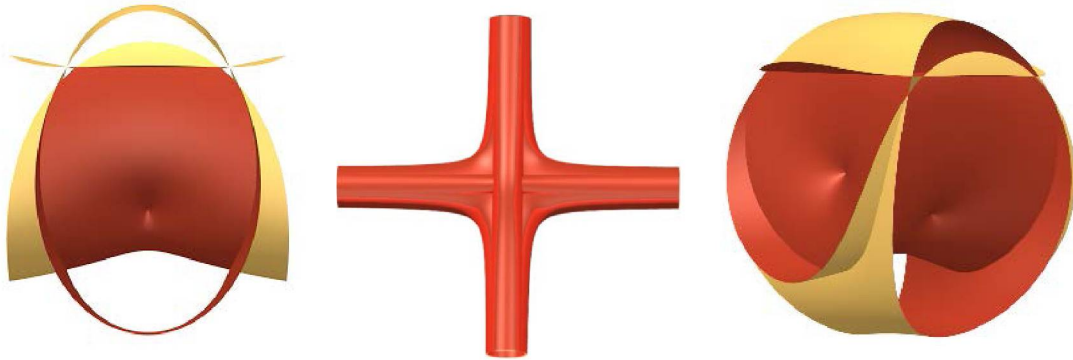


Figure 19: Various views of the Zariski-closure of the 3 –ellipse defined by two perpendicular skew lines and a point on neither of the lines.

The symmetry group of the set of foci, the ellipse, and the Zariski-closure variety is generated by the mirror reflection in yz –plane, the mirror reflection in xy – plane, the 180° rotation about the y –axis.

3 –ellipses with two parallel lines and a point as foci gives us the following two cases.

4.6 Foci are two parallel lines and a point on one of the lines:

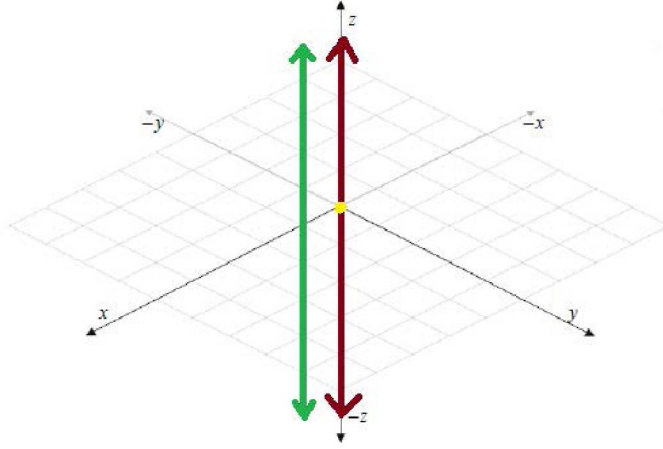


Figure 20: The 3 –foci consisting of two parallel lines and a point on one of the lines.

Consider the parallel lines $L_1 = z\text{-axis}$, $L_2 = \text{line parallel to the } z\text{-axis} = (1, 0, z)$ and the origin $P = (0, 0, 0)$, which is on the line L_1 . To find the algebraic variety V in \mathbb{R}^3 , we choose the $d = 2$ and define the 3 –ellipse as following:

$$E_3 = \{(x, y, z) \in \mathbb{R}^3 : d_1((x, y, z), z\text{-axis}) + d_2((x, y, z), (1, 0, z)) + d_3((x, y, z), (0, 0, 0)) = 2\}.$$

Therefore, we get the equation $\sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + y^2 + z^2} = 2$.

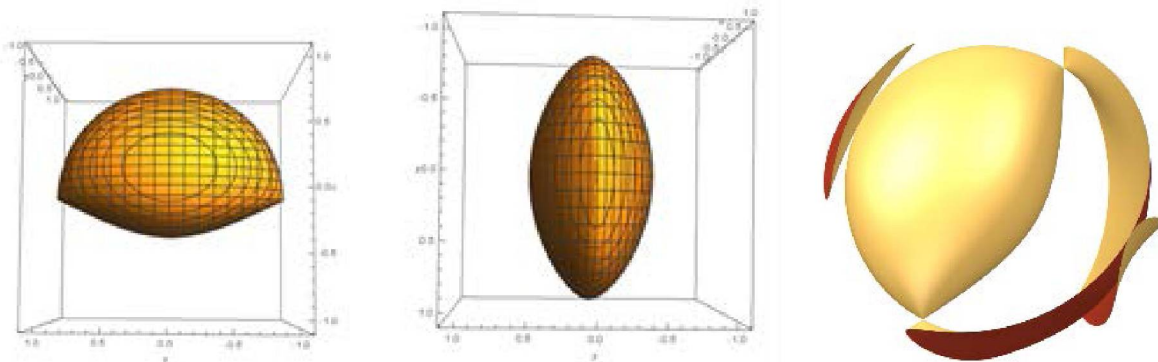


Figure 21: Two views of the 3 –ellipse defined by two parallel lines and a point on one of the lines. The last view shows part of the Zariski-closure as well.

Hence, the polynomial for the Zariski-closure variety becomes (Wolfram Mathematica)

$$\begin{aligned}
 f(x, y, z) = & 81 + 216x - 252x^2 - 456x^3 + 270x^4 + 264x^5 - 108x^6 - 24x^7 + 9x^8 - 468y^2 \\
 & - 552xy^2 + 620x^2y^2 + 496x^3y^2 - 316x^4y^2 - 72x^5y^2 + 36x^6y^2 + 366y^4 \\
 & + 232xy^4 - 308x^2y^4 - 72x^3y^4 + 54x^4y^4 - 100y^6 - 24xy^6 + 36x^2y^6 + 9y^8 \\
 & - 180z^2 - 168xz^2 + 244x^2z^2 + 208x^3z^2 - 76x^4z^2 - 40x^5z^2 + 12x^6z^2 + 228y^2z^2 \\
 & + 176xy^2z^2 - 168x^2y^2z^2 - 80x^3y^2z^2 + 36x^4y^2z^2 - 92y^4z^2 - 40xy^4z^2 \\
 & + 36x^2y^4z^2 + 12y^6z^2 + 118z^4 - 56xz^4 + 12x^2z^4 - 8x^3z^4 - 2x^4z^4 - 12y^2z^4 \\
 & - 8xy^2z^4 - 4x^2y^2z^4 - 2y^4z^4 - 20z^6 + 8xz^6 - 4x^2z^6 - 4y^2z^6 + z^8.
 \end{aligned}$$

The degree of the polynomial is 8 and the number of terms is 55.

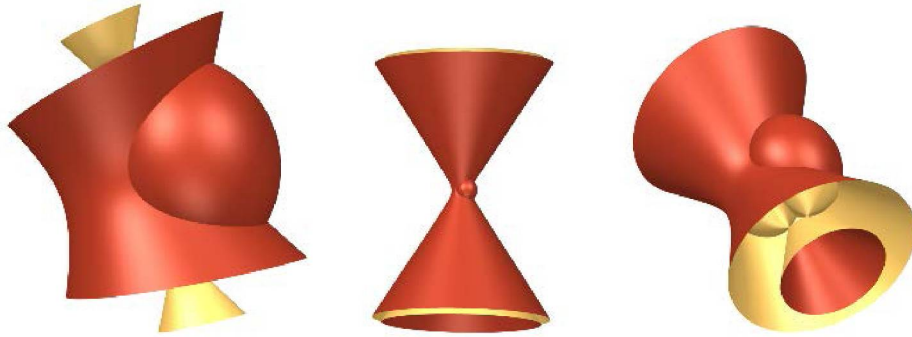


Figure 22: Various views of the Zariski-closure of the 3 –ellipse defined by two parallel lines and a point on one of the lines.

The symmetry group for the of foci, ellipse, and Zariski-closure is generated by the mirror reflection in xz –plane, the mirror reflection in xy –plane and the 180° rotation about the x –axis.

4.7 Two parallel lines and a point on neither of the lines:

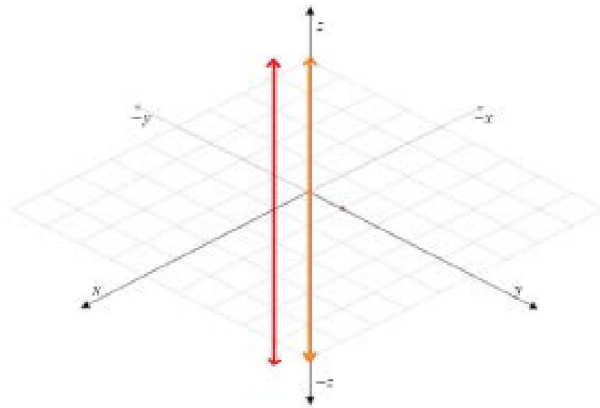


Figure 23: The 3 –foci consisting of two parallel lines and a point on neither of the lines.

Consider the parallel lines $L_1 = z\text{-axis}$, $L_2 = \text{line parallel to the } z\text{-axis} = (1, 0, z)$ and a point $P = (0, 1, 0)$, which is on neither of the lines. To find the algebraic variety V in \mathbb{R}^3 , we choose the $d = 2$ and define the 3-ellipse as following:

$$E_3 = \{(x, y, z) \in \mathbb{R}^3 : d_1((x, y, z), z\text{-axis}) + d_2((x, y, z), (1, 0, z)) + d_3((x, y, z), (0, 1, 0)) = 2\}.$$

Therefore, we get the equation $\sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + (y-1)^2 + z^2} = 2$.

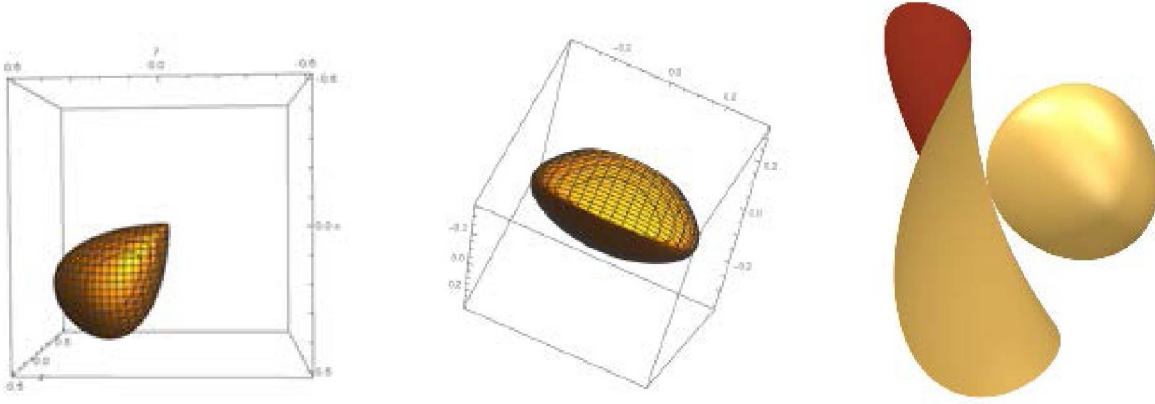


Figure 24: Two views of the 3-ellipse defined by two parallel lines and a point on neither of the lines.

The last view shows part of the Zariski-closure as well.

Hence, the Zariski-closure has the equation (by Wolfram Mathematica)

$$\begin{aligned} f(x, y, z) = & -256x^3 + 192x^4 + 224x^5 - 96x^6 - 24x^7 + 9x^8 + 512xy - 512x^2y - 384x^3y \\ & + 160x^4y + 80x^5y - 24x^6y - 512xy^2 + 448x^2y^2 + 384x^3y^2 - 288x^4y^2 - 72x^5y^2 \\ & + 36x^6y^2 - 256y^3 - 384xy^3 + 384x^2y^3 + 160x^3y^3 - 72x^4y^3 + 192y^4 + 160xy^4 \\ & - 288x^2y^4 - 72x^3y^4 + 54x^4y^4 + 224y^5 + 80xy^5 - 72x^2y^5 - 96y^6 - 24xy^6 \\ & + 36x^2y^6 - 24y^7 + 9y^8 - 256xz^2 + 256x^2z^2 + 192x^3z^2 - 80x^4z^2 - 40x^5z^2 \\ & + 12x^6z^2 - 256yz^2 + 128xyz^2 + 32x^3yz^2 + 8x^4yz^2 + 256xy^2z^2 - 224x^2y^2z^2 \\ & - 80x^3y^2z^2 + 36x^4y^2z^2 + 64y^3z^2 + 32xy^3z^2 + 16x^2y^3z^2 - 144y^4z^2 - 40xy^4z^2 \\ & + 36x^2y^4z^2 + 8y^5z^2 + 12y^6z^2 + 64z^4 - 32xz^4 - 8x^3z^4 - 2x^4z^4 + 96yz^4 \\ & - 48xyz^4 + 24x^2yz^4 - 8xy^2z^4 - 4x^2y^2z^4 + 24y^3z^4 - 2y^4z^4 - 16z^6 + 8xz^6 \\ & - 4x^2z^6 - 8yz^6 - 4y^2z^6 + z^8. \end{aligned}$$

The degree of the polynomial is 8 and number of terms is 75.



Figure 25: Various views of the Zariski-closure of the 3 –ellipse defined by two parallel lines and a point on neither of the lines.

The only symmetry for the set of foci, the ellipse, and the Zariski-closure variety is the mirror reflection in xy – plane.

RESULTS AND OBSERVATION

In this chapter we summarize our observations and state our results.

Observation 5.1: In all seven cases, the foci, the ellipse, and the Zariski-closure variety have the same symmetry groups.

Case 1:

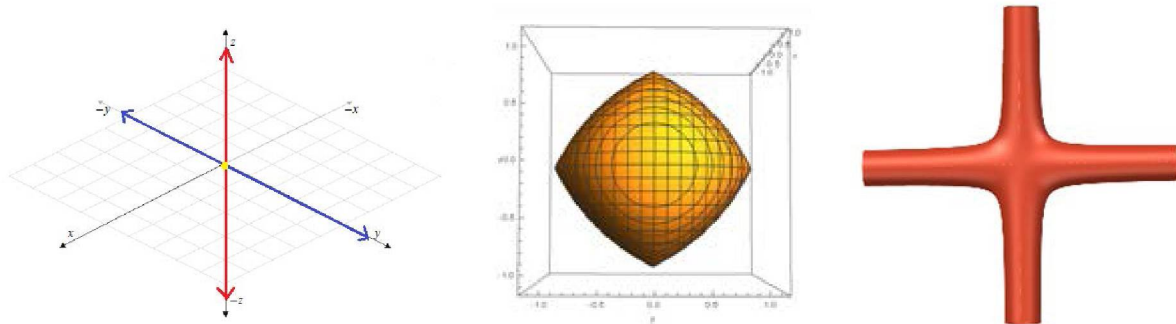


Figure 26: Symmetries of the foci, ellipse, and Zariski-closure: Mirror on the yz – plane, mirror on the xz – plane, and mirror on the xy – plane, three 180° rotations about the x , y and z –axis.

Case 2:

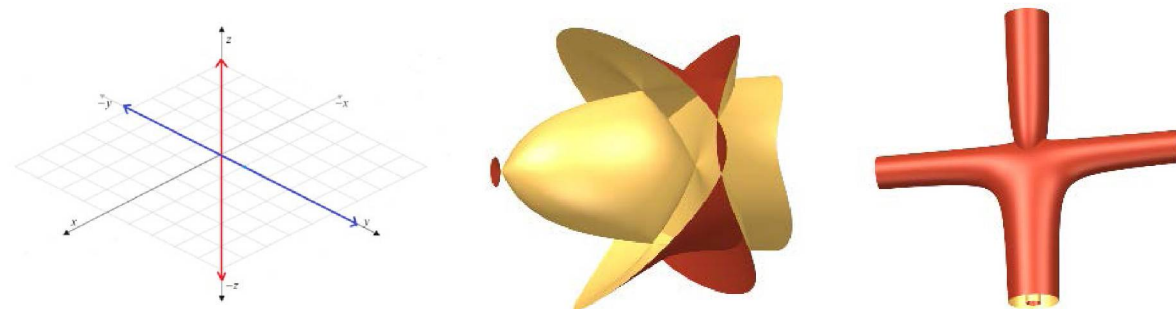


Figure 27: Symmetries of the foci, ellipse, and Zariski-closure: Mirror on the yz – plane, and mirror on the xy – plane, one 180° rotation about the y –axis.

Case 3:

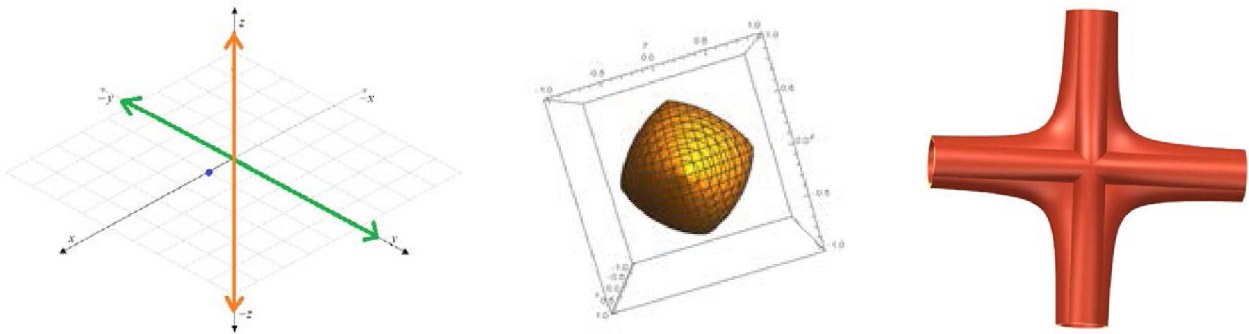


Figure 28: Symmetries of the foci, ellipse, and Zariski-closure: Mirror on the xz – plane, mirror on the xy –plane, one 180° rotation about the x –axis.

Case 4:

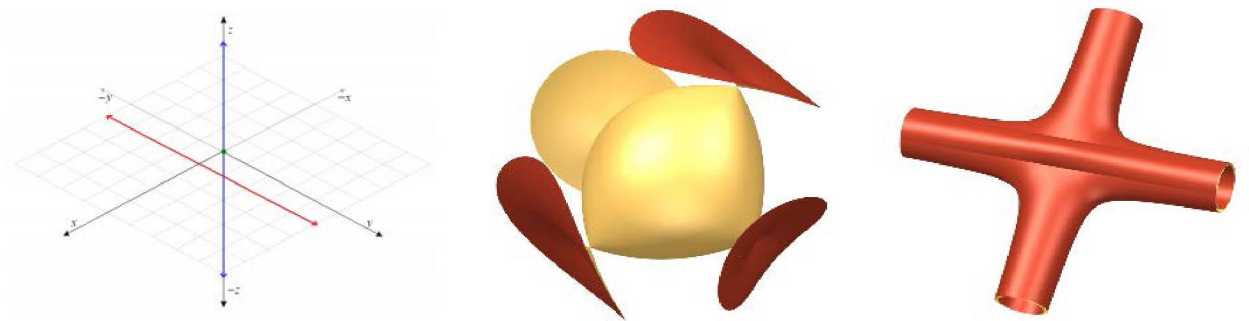


Figure 29: Symmetries of the foci, ellipse, and Zariski-closure: Mirror on the xz – plane, and mirror on the xy –plane, one 180° rotation about the x –axis.

Case 5:

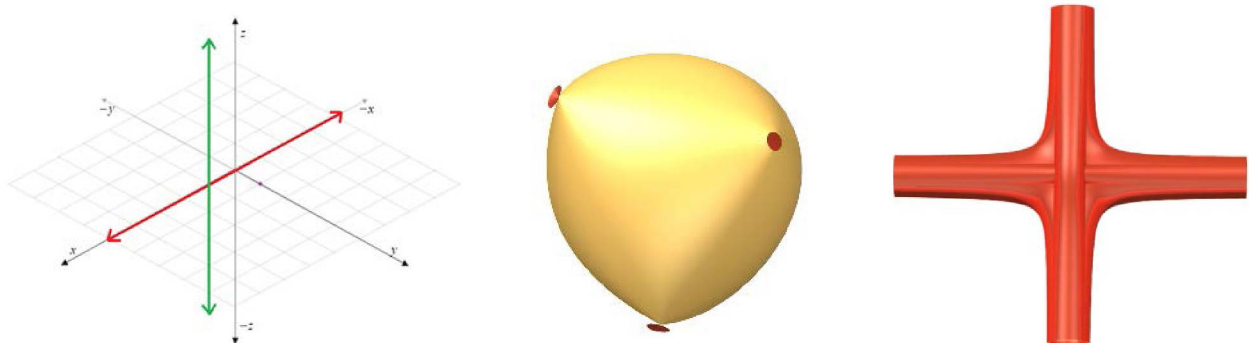


Figure 30: Symmetries of the foci, ellipse, and Zariski-closure: mirror at yz –plane, mirror at xy – plane, one 180° rotation about the y –axis.

Case 6:

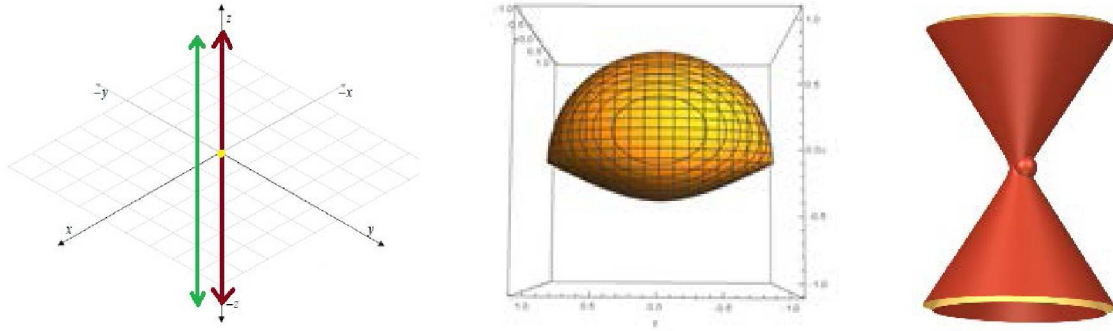


Figure 31: Symmetries of the foci, ellipse, and Zariski-closure: mirror at xz -plane, mirror at xy -plane, one 180° rotation about the x -axis.

Case 7:

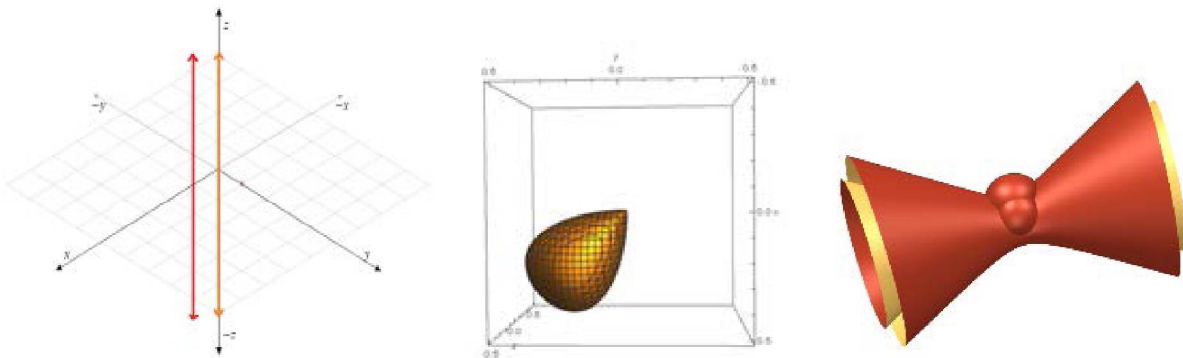


Figure 32: Symmetries of the foci, ellipse, and Zariski-closure: mirror at xy - plane

Note that these findings are consistent with the Theorem 2.2 in Chapter 2 which says that if E is a generalized k -ellipse in \mathbb{R}^n with foci f_1, \dots, f_k (not necessarily points), and if the set of foci has a non-trivial symmetry group G , then E also has the same symmetry group. This observation leads us to the following results:

Lemma 1: Let E be a generalized 3 -ellipse in \mathbb{R}^3 with foci φ_1 , φ_2 and φ_3 that are either points or lines. Then, E has a reflectional symmetry in a plane if and only if the Zariski-closure of E also has the same reflection.

Proof: Let E be a generalized 3-ellipse in \mathbb{R}^3 with foci φ_1, φ_2 and φ_3 that are either lines or points. E is given by $E = \{(x, y, z) \in \mathbb{R}^3 \mid d((x, y, z), \varphi_1) + d((x, y, z), \varphi_2) + d((x, y, z), \varphi_3) = d\}$. Moreover, let Z_E be the Zariski-closure variety of E .

(\Rightarrow) Without loss of generality, suppose E has a reflection in the xy -plane. We want to show that Z_E also has a reflection in the xy -plane. In other words, if $f(x, y, z) = f(x, y, -z)$, where f is the equation of the 3-ellipse E , then $F(x, y, z) = F(x, y, -z)$, where F is the equation of the Zariski-closure variety of E .

Now suppose, $f(x, y, z) = f(x, y, -z)$ and let $f(x, y, z) = A + B + C - d = 0$ as before (Chapter 2). Consider the points (x_0, y_0, z) and $(x_0, y_0, -z)$ on E , where x_0, y_0 are fixed. For the points (x_0, y_0, z) , using variable z the equation can be written as follows

$$A(z) + B(z) + C(z) - d = 0.$$

$$\Rightarrow \sqrt{C_A + (z - a)^2} + \sqrt{C_B + (z - b)^2} + \sqrt{C_C + (z - c)^2} - d = 0, \text{ where } C_A, C_B, C_C \text{ are constants.}$$

Note that because all the terms above are positive (i.e. nothing can cancel) and $f(x, y, z) = f(x, y, -z)$,

$$\begin{aligned} \sqrt{C_A + (z - a)^2} + \sqrt{C_B + (z - b)^2} + \sqrt{C_C + (z - c)^2} - d &= 0 \\ \Rightarrow \sqrt{C_A + z^2} + \sqrt{C_B + z^2} + \sqrt{C_C + z^2} - d &= 0. \end{aligned}$$

Similarly, for the points $(x_0, y_0, -z)$, the equation also will be

$$\sqrt{C_A + (-z)^2} + \sqrt{C_B + (-z)^2} + \sqrt{C_C + (-z)^2} - d = 0.$$

Now we multiply both of these equations by their 7 different appropriate conjugates (See Chapter 4 for more details), and obtain the factored polynomials for the Zariski-closure algebraic varieties, using radicals. Since this factorization is unique, we have that $F(x, y, z) = F(x, y, -z)$; i.e. Z_E is also has a reflection with respect to the xy -plane.

(\Leftarrow) Without loss of generality, suppose Z_E has a reflection in the xy -plane. Consider the points (x_0, y_0, z) and $(x_0, y_0, -z)$ on E . Because Z_E contains E , (x_0, y_0, z) and $(x_0, y_0, -z)$ are also on Z_E . Since Z_E has a reflection in the xy -plane, $F(x_0, y_0, z) = F(x_0, y_0, -z)$. This implies,

$$\begin{aligned} &(A(z) + B(z) + C(z) - d)(A(z) - B(z) + C(z) - d)(A(z) + B(z) - C(z) - d)(-A(z) + B(z) + C(z) \\ &\quad - d)(A(z) - B(z) - C(z) - d)(-A(z) - B(z) + C(z) - d)(-A(z) + B(z) - C(z) - d) \\ &= (A(-z) + B(-z) + C(-z) - d)(A(-z) - B(-z) + C(-z) - d)(A(-z) + B(-z) \\ &\quad - C(-z) - d)(-A(-z) + B(-z) + C(-z) - d)(A(-z) - B(-z) - C(-z) - d)(-A(-z) \\ &\quad - B(-z) + C(-z) - d)(-A(-z) + B(-z) - C(-z) - d) \end{aligned}$$

Since we have unique factorization into the radical expressions related to the distance function, the above equation can only be true when the first factors (positive ones) are equal. Hence,

$$\begin{aligned}(A(z) + B(z) + C(z) - d) &= (A(-z) + B(z) + C(-z) - d). \\ \Rightarrow f(x, y, z) &= f(x, y, -z) \text{ for any } (a, b, z) \in E.\end{aligned}$$

Therefore, E has a reflectional symmetry in the xy -plane. \square

Lemma 2: Let E be a generalized 3 -ellipse in \mathbb{R}^3 with foci φ_1, φ_2 and φ_3 that are either points or lines. Then E has a rotational symmetry around an axis L if and only if the Zariski-closure of E also has the same rotational symmetry around L .

Proof: Let E be a generalized 3 -ellipse in \mathbb{R}^3 with foci φ_1, φ_2 and φ_3 that are either points or lines. Moreover, let Z_E be the Zariski-closure of E .

Without loss of generality, suppose E has a α° rotational symmetry with respect to the z -axis; call it R_z .

We want to show that Z_E also has the same rotational symmetry with respect to the z -axis.

Without loss of generality, suppose θ is an angle on the xy -plane measured from the positive x -axis, and let r be the radial distance from the center. Then by the polar coordinates we have $x = r \cos \theta$ and $y = r \sin \theta$.

Now let $(\theta + \alpha)$ be another angle on the xy -plane measured from the positive x -axis.

Since R_z is a α° rotational symmetry, suppose, $f(r \cos \theta, r \sin \theta, 0) = f(r \cos(\theta + \alpha), r \sin(\theta + \alpha), 0)$, where f is the equation of the 3-ellipse E . We want to show that $F(r \cos \theta, r \sin \theta, 0) = F(r \cos(\theta + \alpha), r \sin(\theta + \alpha), 0)$, where F is the equation of the Zariski-closure i.e. the algebraic variety of E .

Note that $f(r \cos \theta, r \sin \theta, 0) = f(r \cos(\theta + \alpha), r \sin(\theta + \alpha), 0)$ will only be true if either all $a_i = 0$ or $r \cos \theta$ and all $b_i = 0$ or $r \sin \theta$. Then rotating θ to 0, we obtain:

$$\begin{aligned}f(r \cos 0, r \sin 0, 0) &= f(r \cos(\alpha), r \sin(\alpha), 0) \Leftrightarrow f(r, 0, 0) = f(r \cos(\alpha), r \sin(\alpha), 0) \\ \Leftrightarrow \sqrt{r^2 + 0^2 + (-c_1)^2} + \sqrt{r^2 + 0^2 + (-c_2)^2} + \sqrt{r^2 + 0^2 + (-c_3)^2} - d \\ &= \sqrt{r^2 \cos^2 \alpha + r^2 \sin^2 \alpha + (-c_1)^2} + \sqrt{r^2 \cos^2 \alpha + r^2 \sin^2 \alpha + (-c_2)^2} \\ &\quad + \sqrt{r^2 \cos^2 \alpha + r^2 \sin^2 \alpha + (-c_3)^2} - d \\ \Leftrightarrow \sqrt{r^2 + c_1^2} + \sqrt{r^2 + c_2^2} + \sqrt{r^2 + c_3^2} - d &= \sqrt{r^2 + c_1^2} + \sqrt{r^2 + c_2^2} + \sqrt{r^2 + c_3^2} - d.\end{aligned}$$

Now since both sides of this equation are the same, when we multiply each side by 7 different appropriate conjugates (see Chapter 4 for more details), we obtain algebraic varieties that are also the same.

This implies, $F(r, 0, 0) = F(r \cos(\alpha), r \sin(\alpha), 0) \Leftrightarrow f(r \cos 0, r \sin 0, 0) = f(r \cos(\alpha), r \sin(\alpha), 0)$

Rotating back to θ we get that $F(r \cos \theta, r \sin \theta, 0) = F(r \cos(\theta + \alpha), r \sin(\theta + \alpha), 0)$.

Since the xy -plane was an arbitrary choice we conclude that

$$F(r \cos \theta, r \sin \theta, z) = F(r \cos(\theta + \alpha), r \sin(\theta + \alpha), z). \quad \square$$

Lemma 3: Let E be a generalized 3-ellipse in \mathbb{R}^3 with foci φ_1, φ_2 and φ_3 that are either lines points or lines. E is symmetric with respect to the origin if and only if the Zariski-closure of E is also symmetric with respect to the origin.

Proof: Let E be a generalized 3-ellipse in \mathbb{R}^3 with foci φ_1, φ_2 and φ_3 that are either lines or points.

We want to show that if E is symmetric with respect to the origin; in other words, if $f(x, y, z) = f(-x, -y, -z)$, where f is the equation of the 3-ellipse E , then $F(x, y, z) = F(-x, -y, -z)$, where F is the equation of the Zariski-closure i.e. the algebraic variety of E .

Since $f(x, y, z) = f(-x, -y, -z)$, we have

$$\left(\sum_{i=1}^3 \sqrt{(x - a_i)^2 + (y - b_i)^2 + (z - c_i)^2} \right) - d = \left(\sum_{i=1}^3 \sqrt{(-x - a_i)^2 + (-y - b_i)^2 + (-z - c_i)^2} \right) - d.$$

Case 1: Three points: We will have symmetry with respect to the origin when at least one of the 3 –foci is the origin itself and the other two points are positioned where they are symmetric with respect to the origin. For example:

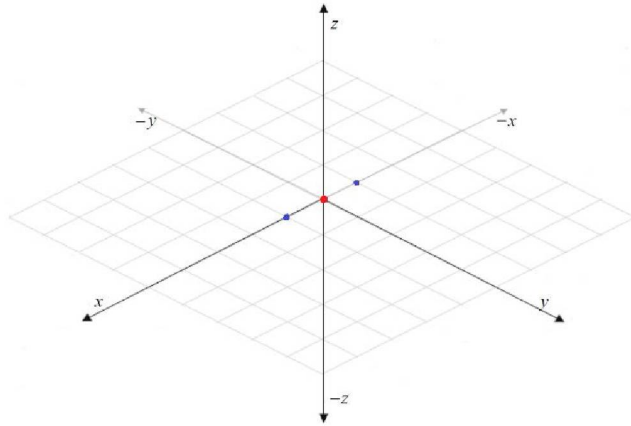


Figure 33: 3 –points foci

So the equation for ellipse may be of the form

$$f(x, y, z) = \left(\sqrt{(x+1)^2 + y^2 + z^2} + \sqrt{x^2 + y^2 + z^2} + \sqrt{(x-1)^2 + y^2 + z^2} \right) - d.$$

Case 2: Two points and one line: We will have symmetry with respect to the origin when the line will go through the origin and the two points will be positioned at a symmetry with respect to the origin. For example:

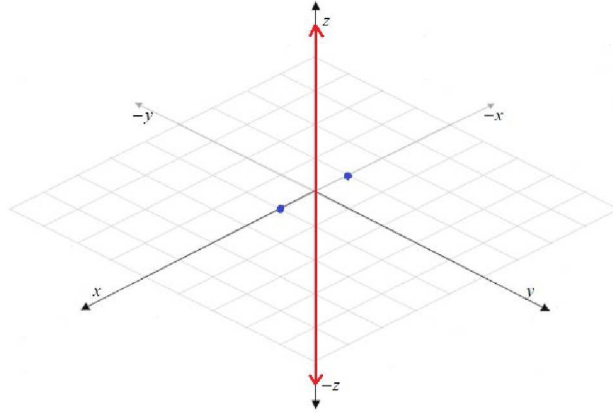


Figure 34: 2 –points and 1 –line foci

So the equation for ellipse may be of the form

$$f(x, y, z) = \left(\sqrt{(x+1)^2 + y^2 + z^2} + \sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + y^2 + z^2} \right) - d.$$

Case 3: One point and two lines: We will have symmetry with respect to the origin when the point is the origin and the two line are symmetric with respect to the origin. For example:

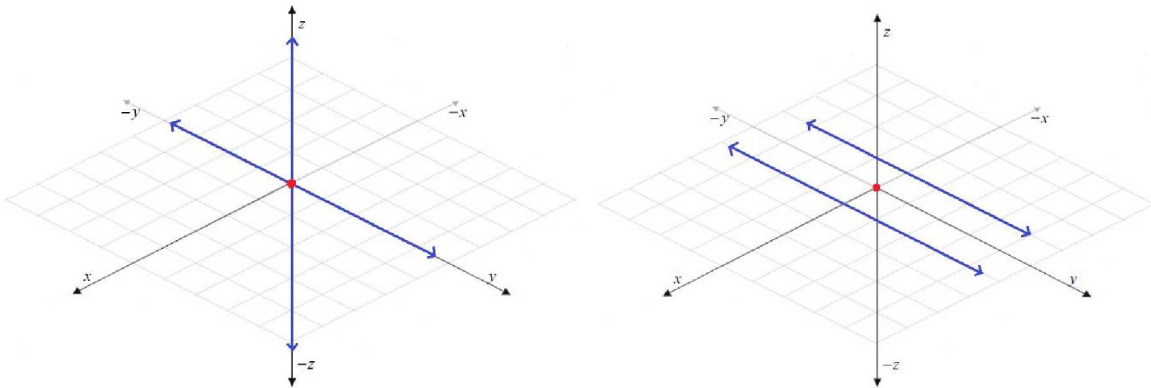


Figure 35: 1 –point and 2 –lines foci

So the equation for ellipse may be of the form

$$f(x, y, z) = \left(\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2 + z^2} + \sqrt{x^2 + z^2} \right) - d$$

Or

$$f(x, y, z) = \left(\sqrt{(x+1)^2 + z^2} + \sqrt{x^2 + y^2 + z^2} + \sqrt{(x-1)^2 + z^2} \right) - d.$$

Case 4: Three lines: We will have symmetry with respect to the origin when at least one of the line goes through the origin and the other two are symmetric with respect to the origin. For example:

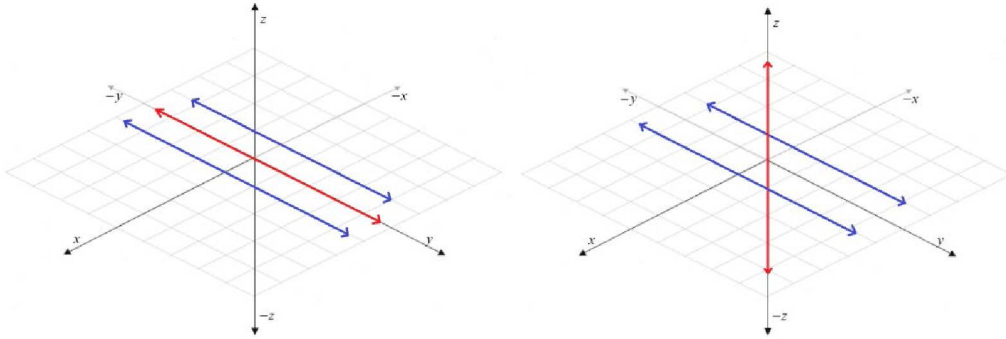


Figure 36: 3 —lines foci

So the equations for ellipse may be of the form

$$f(x, y, z) = \left(\sqrt{(x+1)^2 + z^2} + \sqrt{x^2 + z^2} + \sqrt{(x-1)^2 + z^2} \right) - d$$

or

$$f(x, y, z) = \left(\sqrt{(x+1)^2 + z^2} + \sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + z^2} \right) - d$$

In all the cases above, we can see that $f(x, y, z) = f(-x, -y, -z)$. Now if we multiply both sides by the appropriate conjugates, we can see that it will also be true that $F(x, y, z) = F(-x, -y, -z)$. In other words, the Zariski-closure of the 3 —ellipse will also be symmetric with respect to the origin. \square

Theorem 1: Let E be a generalized 3 —ellipse in \mathbb{R}^3 with foci that are either points or lines. If E has a non-trivial symmetry group G , then the Zariski-closure of E also has the same symmetry group.

Proof: Consequence of the above lemmas. \square

Theorem 2: Let E be a generalized 3 —ellipse in \mathbb{R}^3 with foci $\varphi_1, \varphi_2, \varphi_3$ that are either points or lines. If the set of foci has a non-trivial symmetry group G , then the Zariski-closure of E also has the same symmetry group as the set of foci.

Proof: Let E be a generalized 3 —ellipse in \mathbb{R}^3 with foci $\varphi_1, \varphi_2, \varphi_3$ that are either points or lines. Also let the set of foci have a non-trivial symmetry group G . Then by Theorem 6.4.1 of [17], we know that the 3 —ellipse E has the same symmetry group. However, from the Theorem 1 listed above, we know that whenever a 3 —ellipse E has a non-trivial symmetry group, the Zariski closure of E also has the same symmetry group. Therefore, we conclude that if the set of foci has a non-trivial symmetry group G , the Zariski-closure of E has the same symmetry group as the set of foci. \square

Observation 5.2: In each case, our 3 –ellipse is bounded in \mathbb{R}^3 .

Note that this observation is consisted with Theorem 2.3 (Chapter 2) stating that if E is a generalized k –ellipse in \mathbb{R}^n with foci f_1, \dots, f_k , and if at least one of the foci is a point, then the generalized k –ellipse in \mathbb{R}^n is bounded [17].

Observation 5.3: In all seven cases we considered in the previous chapter, the Zariski-closure varieties are unbounded in \mathbb{R}^3 .

Observation 5.4: In both the intersecting lines cases and the skew lines cases, Zariski-closures have pipe shapes that extends to infinity in 4 ‘directions’. However, in the cases involving parallel lines Zariski-closures have funnel shapes that extends to infinity only in 2 opposite ‘directions’.

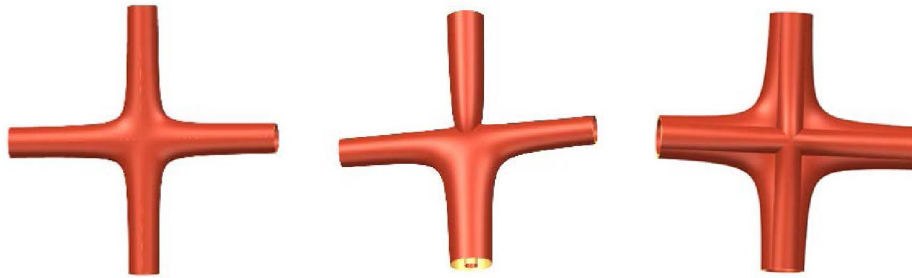


Figure 37: Zariski-closures extend to infinity having pipe shapes in 4 ‘directions’ in the cases of intersecting lines

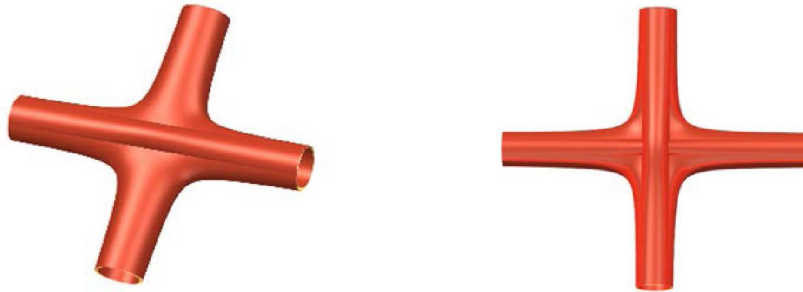


Figure 38: Zariski-closures extend to infinity having pipe shapes in 4 ‘directions’ in the cases of skew lines



Figure 39: Zariski-closures having pipe shapes only in 2 opposite ‘directions’ extend to infinity in the cases of parallel lines

The above observations lead us the following statements.

Theorem 3: Let E be a generalized 3 –ellipse in \mathbb{R}^3 with foci φ_1 , φ_2 and φ_3 consisting of a point and two lines. Then the Zariski-closure variety is not bounded. Moreover,

1. If the two lines are parallel, then the Zariski-closure of the 3-ellipse extends to infinity in the direction of the lines.
2. If the two lines are not parallel, then the Zariski-closure of the 3-ellipse extends to infinity in the directions of both lines.

Proof: Let E be a generalized 3-ellipse in \mathbb{R}^3 with foci φ_1 , φ_2 and φ_3 consisting of a point and two lines.

Proof of 1. Without loss of generality, suppose our L_1 is the x –axis (i.e. $y = 0$ and $z = 0$), L_2 is a line parallel to the x –axis (i.e. $y = 1$ and $z = 0$) and P is the origin. To start, we reduce this case to the case of $\mathbb{R}^2(x, y)$ (i.e. $z = 0$).

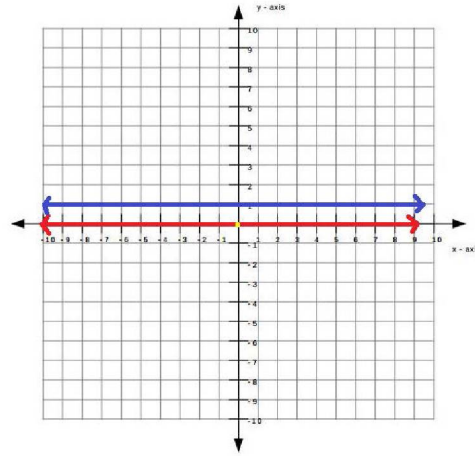


Figure 40: 3-foci consisting two parallel lines and the point of origin in $\mathbb{R}^2(x, y)$

Then for some $d > 1$, the equation of the 3-ellipse becomes $d = |y| + |y - 1| + \sqrt{x^2 + y^2}$.

And therefore, the first step in calculating the equation of the polynomial for the Zariski-closure variety can be expressed as follows:

$$\begin{aligned}
 d - |y| - |y - 1| &= \sqrt{x^2 + y^2} \\
 \Rightarrow (d - |y| - |y - 1|)^2 &= \left(\sqrt{x^2 + y^2}\right)^2 \\
 \therefore d^2 + y^2 + (y - 1)^2 - 2d|y| + 2|y(y - 1)| - 2d|y - 1| &= x^2 + y^2.
 \end{aligned}$$

Note that this equation includes terms with absolute value signs, hence to get the polynomial expression, few more steps are required. However, the zero set for the above equation is included in the Zariski closure variety. Therefore we solve this equation for x without getting it into a polynomial form:

$$\begin{aligned}
x^2 &= d^2 + y^2 + y^2 - 2y + 1 - 2d|y| + 2|y^2 - y| - 2d|y - 1| - y^2 \\
&\Rightarrow x^2 = d^2 + y^2 - 2y + 1 - 2d|y| + 2|y^2 - y| - 2d|y - 1| \\
&\Rightarrow x^2 = 1 + d^2 + y^2 + 2|y^2 - y| - 2|y| - 2d|y| - 2d|y - 1| \\
&\Rightarrow x^2 = 1 + d^2 + y^2 + 2|y^2 - y| - 2|y + dy + d(y - 1)| \\
&\Rightarrow x^2 = 1 + d^2 + y^2 + 2|y^2 - y| - 2|y + 2dy - d| \\
&\Rightarrow x^2 = 1 + d^2 + y^2 + 2|y^2 - y| - 2|y(1 + 2d) - d| \\
&\Rightarrow x^2 = 1 + d^2 + y^2 + 2|y^2 - y| - 2|y(1 + 2d)| + 2d \\
&\Rightarrow x^2 = 1 + d^2 + 2d + y^2 + 2|y^2 - y| - 2|y(1 + 2d)|.
\end{aligned}$$

Note that the right hand side of the above equation is a continuous function of y and when x is large, y has to be large as well. Because d cannot be negative, $1 + d^2 + 2d > 0$. Moreover, $y^2 + 2|y^2 - y| - 2|y(1 + 2d)| \geq 0$ for all $|y| \geq \frac{4}{3}d$. Therefore, for large enough x there exists y such that the above equation has real solutions; i.e. as $x \rightarrow +\infty$, there always exist a real number y that solves the problem above. Moreover, the x^2 term on the left hand side guarantees that the graph of set of zeros of this equation is symmetric. Therefore, the Zariski-closure of this 3-ellipse extends to infinity in both directions along the x -axis.

General case 1. Note that applying a few deformation techniques, we can easily generalize this case to less special situations of 3-ellipses in $\mathbb{R}^3(x, y, z)$. For example we can translate the point from the origin by a distance ϵ in any direction, and obtain equations with similar behavior as $x \rightarrow +\infty$, because the above equation is not going to change too much for large $|x|$ (we may lose the symmetry). Working in this manner, we can conclude that if the two lines of the 3-foci are parallel, then the Zariski-closure of the 3-ellipse extends to infinity along the parallel lines. \square

Proof of 2. Without loss of generality, suppose our L_1 is the x -axis (i.e. $y = 0$ and $z = 0$), L_2 is the y -axis (i.e. $x = 0$ and $z = 0$) and P is the origin. Again, we reduce the study of this case to $\mathbb{R}^2(x, y)$ (assuming $z = 0$).

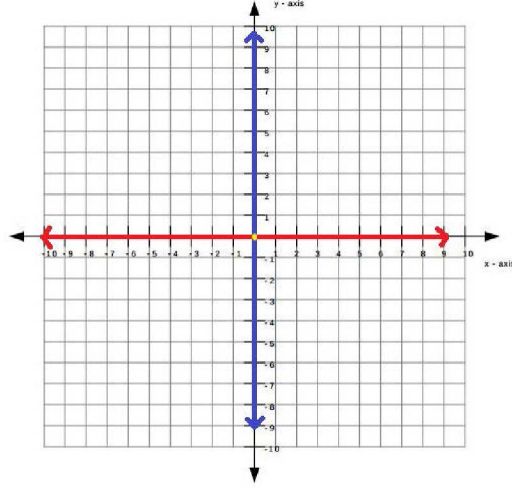


Figure 41: 3 –foci consisting two non-parallel intersecting lines and the point of origin in $\mathbb{R}^2(x, y)$

Then for a fixed distance d , the equation of the 3 –ellipse becomes: $d = |x| + |y| + \sqrt{x^2 + y^2}$.

Therefore, the first step in calculating the equation of the Zariski-closure variety gives us

$$\begin{aligned}
 d - |x| - |y| &= \sqrt{x^2 + y^2} \\
 \Rightarrow (d - |x| - |y|)^2 &= \left(\sqrt{x^2 + y^2}\right)^2 \\
 \Rightarrow d^2 + x^2 + y^2 - 2d|x| + 2|x||y| - 2d|y| &= x^2 + y^2 \\
 \Rightarrow d^2 - 2d|x| + 2|x||y| - 2d|y| &= 0.
 \end{aligned}$$

Note that the solution set is included in the Zariski-closure variety. Hence, we solve for x to study the behavior at infinity.

$$\begin{aligned}
 2|x||y| - 2d|x| &= 2d|y| - d^2 \\
 \Rightarrow |x|(2|y| - 2d) &= 2d|y| - d^2 \\
 \Rightarrow |x| &= \frac{2d|y| - d^2}{2|y| - 2d} \\
 \Rightarrow |x| &= \frac{2d - \frac{d^2}{|y|}}{2 - \frac{2d}{|y|}}.
 \end{aligned}$$

Now, (for all $y \neq d$) as $|y| \rightarrow \infty$, $|x| \rightarrow d$. This means, for large $|y|$ the above equation always has real solutions. Therefore, noting the symmetry due to the $|x|$ term on the left hand side, we conclude that the Zariski-closure variety of this 3 –ellipse extends in both directions along the y –axis.

Next, we solve the equation for y :

$$\begin{aligned}
2|x||y| - 2d|y| &= 2d|x| - d^2 \\
\Rightarrow |y|(2|x| - 2d) &= 2d|x| - d^2 \\
\Rightarrow |y| &= \frac{2d|x| - d^2}{2|x| - 2d} \\
\Rightarrow |y| &= \frac{2d - \frac{d^2}{|x|}}{2 - \frac{2d}{|x|}}.
\end{aligned}$$

Now, (for all $x \neq d$) as $|x| \rightarrow \infty$, $|y| \rightarrow d$. This means that the above equation always has real solutions. Then again, noting the symmetry due to the $|y|$ term on the left hand side, we conclude that the Zariski-closure variety of this 3-ellipse also extends in two opposite directions along the x -axis.

General case 2. Note that applying a few deformation techniques, we can easily generalize this case to other less special 3-ellipses in $\mathbb{R}^3(x, y, z)$. We can translate the point from the origin by a distant of ϵ in any direction, and obtain equations for varieties with similar behavior while tending to infinity. Therefore we conclude that if the two lines are non-parallel, then the Zariski-closure of the 3-ellipse extends to infinity in the directions along the lines as we predicted.

Moreover, also note that deforming the 3-ellipse by moving the line L_2 slightly (translate or rotate), does not significantly change the solutions of the above equations, hence the above can represent behavior of the Zariski closure variety for 3-ellipse in a general case. Therefore, the Zariski-closure variety extends along its linear foci to infinity in all cases. \square

Corollary 1. If the 3-ellipse in \mathbb{R}^3 has a linear foci, then the Zariski-closure variety is unbounded.

Observation 5.5: In all seven cases, the Zariski-closure variety is irreducible because it cannot be written as the union of distinct nonempty algebraic varieties. We can also see this algebraically, because in all seven cases the equations of the algebraic variety are not factorable (as per Wolfram Mathematica). We expect that the generic Zariski-closure varieties for 3-ellipses is irreducible.

Observation 5.6: The degree of the polynomial describing the Zariski-closure variety of all 3-ellipses in all our examples is 8.

This observation leads us to the following results:

Theorem 3: Every 3 –ellipse in $\mathbb{R}^3(x_1, x_2, x_3)$ with foci that are either points or lines has a Zariski-closure of degree ≤ 8 . Moreover, the degree is 8, if the linear foci do not overlap.

Proof: We already know that Zariski-closure of a k –ellipse with k – focal points in general position in \mathbb{R}^n has a polynomial representation of degree 2^k when k is odd [13]. Therefore, every 3 –ellipse with foci consisting of three points has a Zariski-closure V of degree $2^3 = 8$.

Now suppose, one or more of our 3 –foci are lines. Then for some $d > d_w$, $d = A + B + C$, where A, B, C represent the distances from each foci. Hence, we have

$$A + B + C - d = 0.$$

Note that A, B, C are square roots of sums of positive quadratic polynomial terms. Alternating the signs of A, B, C , we multiply the left side of this equation by its seven other conjugates and obtain the following equation (See Chapter 4 for more details):

$$\begin{aligned} P(A, B, C) = & A^8 - 4A^6B^2 + 6A^4B^4 - 4A^2B^6 \\ & + B^8 - 4B^6C^2 + 6B^4C^4 - 4B^2C^6 \\ & + C^8 - 4A^2C^6 + 6A^4C^4 - 4A^6C^2 \\ & + 4A^4B^2C^2 + 4A^2B^4C^2 + 4A^2B^2C^4 \\ & - 4A^6d^2 + 4A^4B^2d^2 + 4A^2B^4d^2 - 4B^6d^2 + 4A^4C^2d^2 - 40A^2B^2C^2d^2 + 4B^4C^2d^2 \\ & + 4A^2C^4d^2 + 4B^2C^4d^2 - 4C^6d^2 + 6A^4d^4 + 4A^2B^2d^4 + 6B^4d^4 + 4A^2C^2d^4 \\ & + 4B^2C^2d^4 + 6C^4d^4 - 4A^2d^6 - 4B^2d^6 - 4C^2d^6 + d^8. \end{aligned}$$

Note that in the above equation, not only the degree of the entire polynomial is 8, but also the highest degree A, B, C individually assume is 8. Moreover, all the powers are even. Hence, there are no radical expressions when we substitute expressions in x_1, x_2, x_3 .

If any two focal lines are in general positions, then the radical expressions in A, B, C are all different. Without the loss of generality we can assume that one of them describes the distance to the x_2 –axis; i.e.

$A = \sqrt{x_1^2 + x_3^2}$. Hence, the first term in the above equation becomes: $x_1^8 + x_3^8 + (\text{lower degree terms})$.

Now if we let, $B = x_1^2 + P_2(x_2, x_3) + P_1(x_1, x_2, x_3)$ and $C = x_1^2 + Q_2(x_2, x_3) + Q_1(x_1, x_2, x_3)$. Then writing only the highest degree terms for x_1 in the above polynomial we have, $P(A, B, C) = x_1^8 - 4x_1^8 + 6x_1^8 - 4x_1^8 + x_1^8 - 4x_1^8 + 6x_1^8 - 4x_1^8 + 4x_1^8 + 4x_1^8 + 4x_1^8 + F(x_2, x_3) + G(x_1, x_2, x_3)$ where the exponent of x_1 is less than 8. Considering only the coefficients of x_1^8 we

obtain $1 - 4 + 6 - 4 + 1 - 4 + 6 - 4 + 1 - 4 + 6 - 4 + 4 + 4 + 4 \neq 0$. This means x_1^8 does not cancel; i.e. the polynomial is of degree 8 when each of A, B, C contains the term x_1^2 .

Now let us consider the case when B or C does not have any x_1^2 term. Without the loss of generality, suppose it is C . Note that at least two of x_1, x_2, x_3 must be present in any distance expression in \mathbb{R}^3 . Hence we let $B = x_1^2 + x_2^2 + (\text{lower degree terms})$ and $C = x_2^2 + x_3^2 + (\text{lower degree terms})$.

Then writing only the highest degree terms only in the above polynomial we obtain the following equation

$$\begin{aligned} P(A, B, C) = & (x_1^8 + x_3^8 + \dots) - 4(x_1^6 + x_3^6 + \dots)(x_1^2 + x_2^2 + \dots) + 6(x_1^4 + x_3^4 + \dots)(x_1^4 + x_2^4 + \dots) - \\ & 4(x_1^2 + x_3^2 + \dots)(x_1^6 + x_2^6 + \dots) + (x_1^8 + x_2^8 + \dots) - 4(x_1^6 + x_2^6 + \dots)(x_2^2 + x_3^2 + \dots) + 6(x_1^4 + \\ & x_2^4 + \dots)(x_2^4 + x_3^4 + \dots) - 4(x_1^2 + x_2^2 + \dots)(x_2^6 + x_3^6 + \dots) + (x_2^8 + x_3^8 + \dots) - 4(x_1^2 + x_3^2 + \\ & \dots)(x_2^6 + x_3^6 + \dots) + 6(x_1^4 + x_3^4 + \dots)(x_2^4 + x_3^4 + \dots) - 4(x_1^6 + x_3^6 + \dots)(x_2^2 + x_3^2 + \dots) + 4(x_1^4 + \\ & x_3^4 + \dots)(x_1^2 + x_2^2 + \dots)(x_2^2 + x_3^2 + \dots) + 4(x_1^2 + x_3^2 + \dots)(x_1^4 + x_2^4 + \dots)(x_2^2 + x_3^2 + \dots) + 4(x_1^2 + \\ & x_3^2 + \dots)(x_1^2 + x_2^2 + \dots)(x_2^4 + x_3^4 + \dots) + \dots. \end{aligned}$$

In this case, the coefficients of x_1^8, x_2^8, x_3^8 may be zero, however the coefficients for the final term $x_1^6 x_2^2$ is not. Hence, the polynomial is of degree 8. We can do similar calculations in general cases. Therefore, we can conclude that the degree of the Zariski-closure of every 3-ellipse in \mathbb{R}^3 is at most 8, and is equal to eight for generic cases. \square

Theorem 4. Family of polynomials in three variables of *degree* ≤ 8 can be classified by \mathbb{P}^{164} .

Proof of claim 2. We know that a parameter space in this case is the set of the possible coefficients of polynomials in three variables of degree 8 that are not all simultaneously zero. We can calculate the dimension of the space parameterizing all homogeneous polynomials of degree eight in $\mathbb{P}^3(x, y, z, w)$ as follows. The dimension of the parameter space is given by $\binom{8+3}{8} - 1 = 164$.

Therefore, all polynomials of degree 8 with 3 variables are parameterized by \mathbb{P}^{164} . Hence, we conclude that the subset containing varieties that are Zariski-closures of 3-ellipses is of dimension at most 164 (but most likely less than 164) in \mathbb{P}^{164} . \square

Corollary 3: The set of polynomials for the Zariski-closure varieties of 3-ellipses consisting can be considered as a subset in \mathbb{P}^{164} with some embedding.

One can ask which polynomials of degree 8 represent Zariski closure varieties of 3-ellipses and how to characterize them.

COMPARISON

In this short chapter, we compare our results with the statements mentioned in Chapter 1.

Theorem: Zariski-closure of a k –ellipse consisting of k points in \mathbb{R}^n has a polynomial representation of degree 2^k when k is odd, and degree $2^k - \binom{k}{k/2}$ when k is even [13].

Comparison: This result extends to lines and points in our study. In all seven cases, the polynomial of the Zariski-closure variety of the 3 –ellipse has a polynomial representation of degree $2^3 = 8$ and the result extends to the generic case with similar foci.

Theorem: Let E be a generalized k –ellipse in \mathbb{R}^n with foci f_1, \dots, f_k (not necessarily points). If the set of foci has a non-trivial symmetry group G , then E also has the same symmetry group [17].

Comparison: Through our study, we have extended this result in our cases for 3 –ellipses to include the Zariski-closure varieties V . Hence we expect the following to be true in general:

Let E be a generalized k –ellipse in \mathbb{R}^n with foci f_1, \dots, f_k (not necessarily points). If the set of foci has a non-trivial symmetry group G , then E and V also has the same symmetry group

Theorem: Let E be a generalized k –ellipse in \mathbb{R}^n with foci f_1, \dots, f_k . If at least one of the foci is a point, then the generalized k –ellipse in \mathbb{R}^n is bounded [17].

Comparison: All seven of our cases had exactly one point as a focus and each one of our 3 –ellipses is bounded. Therefore, this result has been verified.

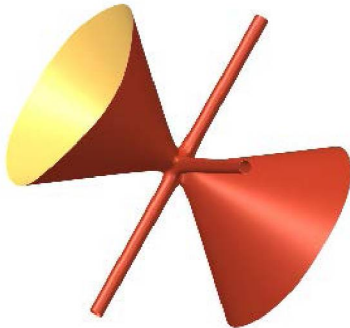
Deformations

Our 7 cases are very specific examples 3 –ellipses and can be easily deformed to general examples by simple rotations and translations of the foci. As an example, now we analyze some deformations of one of equations of the Zariski-closure variety, namely Case 5 in Chapter 5:

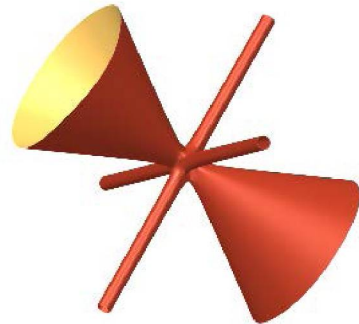
$$\begin{aligned}
 &36864 - 24576x^2 + 4096x^4 - 33280y^2 + 7168x^2y^2 - 512x^4y^2 + 7952y^4 - 992x^2y^4 + \\
 &16x^4y^4 - 520y^6 + 24x^2y^6 + 9y^8 - 27136z^2 + 5120x^2z^2 - 512x^4z^2 - 3072yz^2 + \\
 &1024x^2yz^2 + 10912y^2z^2 - 1344x^2y^2z^2 + 32x^4y^2z^2 - 704y^3z^2 + 64x^2y^3z^2 - 968y^4z^2 + \\
 &56x^2y^4z^2 + 48y^5z^2 + 24y^6z^2 + 3600z^4 - 480x^2z^4 + 16x^4z^4 - 960yz^4 + 64x^2yz^4 - \\
 &416y^2z^4 + 32x^2y^2z^4 + 64y^3z^4 + 16y^4z^4 = 0.
 \end{aligned}$$

Note that the term $+9y^8$ has the highest degree in one variable and it is also the only term with an odd coefficient. Here we study the stability of the properties of our Zariski closure surface under deformations by continuously changing the coefficient of the term ay^8 and observing graphs below.

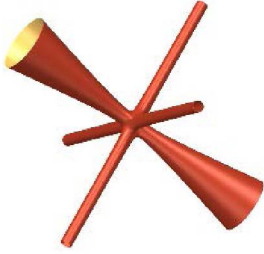
When $a = -20$:



When $a = -9$:



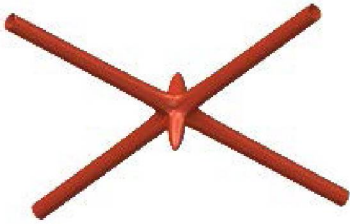
When $a = -1$:



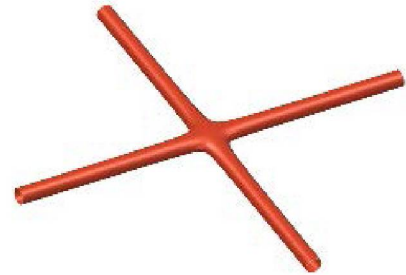
When $a = 0$:



When $a = 1$:



When $a = 9$: (our usual case)



When $a = 11$:



When $a = 20$:



Figure 42. A family of interesting deformations

It can be easily observed that as the coefficient of the term y^8 goes to $-\infty$, we obtain a funnel-type shape which gets bigger and bigger. On the contrary, as the coefficient of the term y^8 goes to ∞ , the double-red-cross vanishes on itself in the middle.

CONCLUSIONS

Most of our results are based on properties we observed while graphing the 3-ellipses and their associated algebraic varieties. Hence, they may not seem very surprising. However, we are excited to see that our findings are compatible with other known results. It is interesting to observe that the 3-ellipses and their Zariski-closures always have the same number and types of symmetries. We would like to predict that this result can be extended to not only lines and planes, but also to any flats and even to higher dimensions. Moreover, for all the cases of 3-foci consisting two lines and a point, we obtain polynomials for Zariski-closure algebraic varieties that have at most 75 terms. Therefore, it may be safe to say that such Zariski-closure varieties of 3-ellipses may be classified by some suitable parameter space of dimension 74 (that is less than the dimension 164 of the total parameter space of degree 8 polynomials). One can ask which polynomials of degree 8 represent Zariski-closure varieties of 3-ellipses and how to characterize them. There should be more work done on other properties of Zariski-closure algebraic varieties, including their irreducibility, singularities and topological properties.

BIBLIOGRAPHY

- [1] "Algebraic Variety." Algebraic Variety -- from Wolfram MathWorld. Wolfram MathWorld, n.d. Web. 14 Apr. 2017.
- [2] Circle with Segments. Digital image. Wikimedia.org, n.d. Web. 26 Feb. 2018.
<<https://upload.wikimedia.org/wikipedia/commons/thumb/0/03/Circle-withsegments.svg/1200px-Circle-withsegments.svg.png>>.
- [3] "Dimension of an Algebraic Variety". *En.wikipedia.org*. N.p., 2017. Web. 14 Apr. 2017.
- [4] Ellipsoid. Digital image. Technologyuk.net, n.d. Web. 26 Feb. 2018.
<https://www.technologyuk.net/mathematics/geometry/images/geometry_0203.gif>.
- [5] Foci of an Ellipse. Digital image. Softschools.com, n.d. Web. 26 Feb. 2018.
<http://www.softschools.com/math/calculus/images/finding_the_foci_of_an_ellipse_img_2.png>.
- [6] Grzegorczyk, I. "Algebraic Geometry Handout 1". (2017): 8. Print.
- [7] Grzegorczyk, I. "Algebraic Geometry Handout 2". (2017): 2-3. Print.
- [8] Grzegorczyk, I., Elliot J., Tejeda K., Stebbins J., Jasso T. "Generalized k-ellipses and Their Closures". (2017): 18-23. PowerPoint slides.
- [9] Grzegorczyk, I., Suarez R. "Examples of the Zariski Closure of 3-ellipses". (2017): 1. PDF file.
- [10] Hulek, K. "Elementary Algebraic Geometry". Providence, RI: American Mathematical Society, 2003. Print.
- [11] "Irreducible Variety -- From Wolfram Mathworld". *Mathworld.wolfram.com*. N.p., 2017. Web. 14 Apr. 2017.
- [12] Jasso, T. "Zariski Closures of Generalized Ellipses" (Master's thesis). (2017). California State University Channel Islands, Camarillo, CA.
- [13] Nie, J., Parrilo, P. A., & Sturmfels, B. "Semidefinite Representation of the k-Ellipse". (2008): *Algorithms in Algebraic Geometry the IMA Volumes in Mathematics and its Applications*, 117-132. doi:10.1007/978-0-387-75155-9_7
- [14] Skew Line. Digital image. Onlinemathlearning.com, n.d. Web. 26 Feb. 2018.
<<https://www.onlinemathlearning.com/image-files/xlines.png.pagespeed.ic.PSZ-Umrl86.png>>.

- [15] Sphere and Ball. Digital image. Wikimedia.org, n.d. Web. 26 Feb. 2018.
https://upload.wikimedia.org/wikipedia/commons/thumb/0/07/Sphere_and_Ball.png/220px-Sphere_and_Ball.png.
- [16] Stebbins, J. "Generalized Ellipses in \mathbb{R}^n with Multidimensional Foci" (Unpublished master's thesis). (2014). California State University Channel Islands, Camarillo, CA.
- [17] Tejada, K. "Cataloging Generalized K-Ellipses" (Unpublished master's thesis). (2015). California State University Channel Islands, Camarillo, CA.
- [18] Trachsler, B., & Guggisberg, M. (n.d.). "Solving the Fermat-Weber Problem a Numerical and Geometric Approach". Retrieved November 26, 2017, from
https://mgje.github.io/presentations/Budapest2014/slides_Fermat-Weber.pdf
- [19] Vincze, P. E. (1982). "On the approximation of convex, closed plane curves by multifocal ellipses". *Journal of Applied Probability*, 19(A), 89-96. doi:10.2307/3213552