Trampoline Graphs and Estimates of Their Pebbling Numbers

By

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Trampoline Graphs and Estimates of Their Pebbling Numbers
Title of Item

Graph Pebbling, Pebbling Number, Trampoline Graph, Unsolvable Distribution
3 to 5 keywords or phrases to describe the item

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Dedication

I dedicate this to my wonderful fiancé, Drew, for his love and support through my long hours of studying and constant graph sketching. I also dedicate this to my family for always believing in me and continually encouraging me to strive for bigger things.

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Camarillo, California

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ABSTRACT

Graph pebbling is a mathematical game in which pebbles are placed on the vertices of a graph. The game is made up of a series of pebbling steps that consist of removing two pebbles from one vertex, discarding a pebble, and placing the other on an adjacent vertex. The goal of the game is to reach a particular vertex by performing a series of pebbling steps. This paper will focus on implementing particular pebbling strategies on specific types of graphs to determine the minimum amount of pebbles needed to reach a vertex via pebbling moves. We will use these strategies as a basis for determining the lower bound of the pebbling number of our graph.

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CHAPTER 1

INTRODUCTION TO PEBBLING

Graph pebbling is a mathematical game in which pebbles are placed on the vertices of a graph G. The game is made up of a series of pebbling steps. A pebbling step consists of removing two pebbles from one vertex, discarding one, and placing the other on an adjacent vertex. The goal of the game is for a pebble to be moved to a particular vertex T, the target vertex, by performing a series of pebbling steps.

If D is a distribution of pebbles onto the vertices of G that allows us to move a pebble to the target vertex T, then we say that D is *T-solvable*; otherwise, D is *T-unsolvable*. D is *solvable* if it is *T-solvable* for all T, and *unsolvable* otherwise. $D(v)$ is the number of pebbles on vertex v in D, where $|D|$ is the total number of pebbles in D such that $|D| = \sum_v D(v)$. A *configuration* is a function on G from the vertices of G to the nonnegative integers. The terms distribution and configuration will be used interchangeably throughout this thesis.

The *pebbling number* of a graph G, $\pi(G)$, is the lowest natural number n such that for any distribution of n pebbles on the vertices of G, one pebble can be moved to any specified target vertex T. Very little is known about the pebbling number except in the case of very particular classes of graphs such as paths, cycles, trees, and complete graphs [2].

Let P_n be a path with vertex set $V(P_n) = \{v_0, v_1, ..., v_{n-1}\}$ and edge set $E(P_n) = \{v_i v_{i+1} | i = 0, 1, ..., n-2\}$, where $|V(P_n)| = n$.

Example 1.0.1 *Consider P_7 with $D(v_6) = 64$ and $D(v_i) = 0$, $\forall\ v_i \in V(P_7) - \{v_6\}$. Can we reach $T = v_0$ with this distribution of pebbles on P_7?*

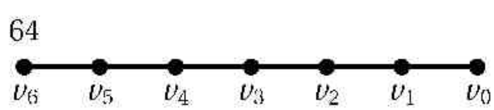


Figure 1.1: P_7 with $D(v_6) = 64$ and $D(v_i) = 0$, $\forall\ v_i \in V(P_7) - \{v_6\}$.

Figure 1.2 illustrates how we can reach T. We will explain next in further detail.

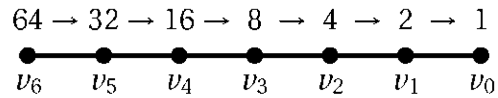


Figure 1.2: P_7 with $D(v_6) = 64$ and pebbling moves to reach v_0.

Start with 64 pebbles on vertex v_6. Send 32 pebbles to v_5, discarding 32 at the same time. Send 16 pebbles to v_4, discarding 16 at the same time. Send 8 pebbles to v_3, discarding 8 at the same time. Send 4 pebbles to v_2, discarding 4 at the same time. Send 2 pebbles to v_1, discarding 2 at the same time. Send 1 pebble to v_0, discarding 1 at the same time. Therefore, we can reach v_0 when $D(v_6) = 64$.

Example 1.0.2 *This time, let $D(v_6) = 63$ and $D(v_i) = 0$, $\forall\ v_i \in V(P_7) - \{v_6\}$. Can we reach $T = v_0$ with this configuration of pebbles?*

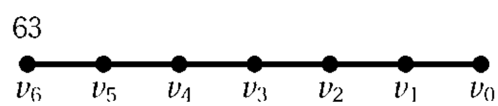


Figure 1.3: Initial configuration of $D(v_6) = 63$ on P_7.

The only way to possibly reach v_0 using pebbles on v_6 is to send all 63 pebbles directly across the path towards v_0, but we will show that this process, given this configuration of pebbles, will not permit us to reach v_0.

Start with 63 pebbles on vertex v_6. Send 31 pebbles to v_5, discard 31, and leave 1 pebble at v_6. Send 15 pebbles to v_4, discard 15, and leave 1 pebble at v_5. Send 7 pebbles to v_3, discard 7, and leave 1 pebble at v_4. Send 3 pebbles to v_2, discard 3, and leave 1 at v_3. Send 1 pebble to v_1, discard 1, and leave 1 at v_2. Hence, we cannot reach v_0 when $D(v_6) = 63$.

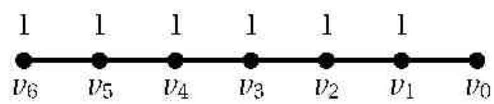


Figure 1.4: Result of attempting to reach v_0 when $D(v_6) = 63$ on P_7.

Therefore, removing even a single pebble from the configuration, i.e. changing $D(v_6) = 64$ to $D(v_6) = 63$ when $D(v_i) = 0,\ \forall\ v_i \in V(P_7) - \{v_6\}$, eliminates our ability to successfully reach v_0, the target vertex of the graph. This means we have found a distribution of 64 pebbles on P_7 that is solvable and a distribution of 63 pebbles that is unsolvable. This implies that $\pi(P_7) \geq 64$.

Moews [4] has proven that the pebbling number of P_n, a path on n vertices, is $\pi(P_n) = 2^{n-1}$. This means that for the graph above, $\pi(P_7) = 2^6 = 64$. Moews' proof utilizes the Weight Argument, which he also developed:

Our ability to reach vertex a_0 on a path P_n is unchanged if we remove one pebble from a_j and place two pebbles on a_{j+1}, where j is the distance from a_j to our target vertex a_0. This is because any pebbling move from a_{j+1} to a_0 would require two pebbles from a_{j+1} to be removed, with one pebble moved back to a_j as we attempt to reach a_0.

If pebbles are placed on the vertices of P_n in such a way that

$$\sum_{i=0}^{n-1} D(a_i)2^{-i} \geq 1,$$

we can pebble a_0. If we sequentially remove one pebble from a_j and place two pebbles on a_{j+1} for j increasing, $0 < j < n-1$, we get

$$\sum_{i=0}^{n-1} D(a_i)2^{1-n}$$

pebbles on a_{n-1}. But by hypothesis, this number is at least 2^{n-1}, meaning we can pebble a_0. Therefore $\pi(P_n) = 2^{n-1}$.

The weight argument can be utilized to prove the pebbling number of a path because paths are *greedy graphs* [2]. Every configuration of size at least $\pi(P_n)$ has a *greedy solution*, meaning that every pebbling step is a step from a vertex v_i to a vertex v_j such that $d(v_j, v_0) < d(v_i, v_0)$, where v_0 is the target vertex. The weight argument provides a strategy for solving pebbling numbers of greedy graphs, a strategy that cannot be utilized for graphs that are not greedy. This suggests that finding the pebbling number of a non-greedy graph can be more complex because fewer applicable tools have been discovered.[3]

Notice that in Examples 1.0.1 and 1.0.2 above, as soon as we found an unsolvable pebbling distribution for our graph, we also discovered a lower bound for the pebbling number of the graph. In general, the pebbling number of a graph contains one more pebble than the maximum t such that there exists an unsolvable pebbling distribution of size t [2]. Therefore, determining an unsolvable distribution for a graph is a useful tool towards determining a lower bound of the pebbling number of a graph, which is the first step in determining the pebbling number. This will be examined in further detail in future chapters.

Chapter 2

Introduction To The Trampoline Graph

In this chapter we will provide examples that demonstrate the use of a particular pebbling strategy on a specific graph. This will help us to develop the concept of the trampoline graph.

Example 2.0.3 *Let* $R = P_7 = \{v_0, v_1, ..., v_6\}$ *be the same path as before. Let* $A = \{u_1, u_2, u_3, u_4\}$ *be an entirely new arch such that our new graph has vertex set* $R \cup A$ *and edge set* $\{v_i v_{i+1} | i = 0, 1, ..., 5\} \cup \{u_i u_{i+1} | i = 1, 2, 3\} \cup \{v_1 u_1\} \cup \{u_4 v_5\}$. *Define* $D : V(P_7) \longrightarrow Z$ *by* $D(v_6) = 48$ *and* $D(v_i) = 0, \forall\ v_i \in V(P_7) - \{v_6\}$. *Define* $D : V(A) \longrightarrow Z$ *by* $D(u_2) = 3$ *and* $D(u_j) = 0, \forall\ u_j \in V(A) - \{u_2\}$. *Can we reach* v_0

with this configuration of pebbles?

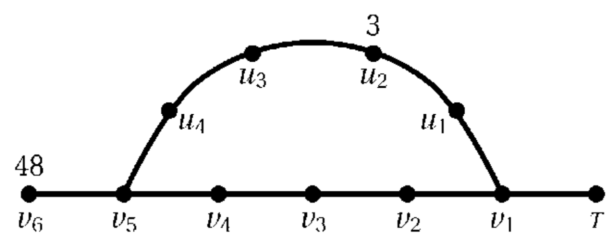


Figure 2.1: The configuration described in Example 2.0.3.

First, we note that if $D(v_6) = 48$, we will not reach v_0 if we send all pebbles from v_6 across P_7 (since previous examples have shown that we need $D(v_6) \geq 64$ to do so). We also note that if $D(u_2) = 3$, we can only reach either u_1 or u_3 with these 3 pebbles. This means we cannot reach any of the vertices on P_7, therefore these 3 pebbles cannot help us to reach v_0 when all pebbles at v_6 are sent across P_7. Then we cannot reach v_0 by sending the 48 pebbles at v_6 directly across P_7.

What if we instead try to send all of the pebbles at v_6 up the arch A? Send 24 pebbles to v_5 and discard 24. Send 12 pebbles to u_4 and discard 12. Send 6 pebbles to u_3 and discard 6. Send 3 pebbles to u_2 and discard 3. We can now add these 3 pebbles to the 3 pebbles already sitting at u_2. Send 3 pebbles to

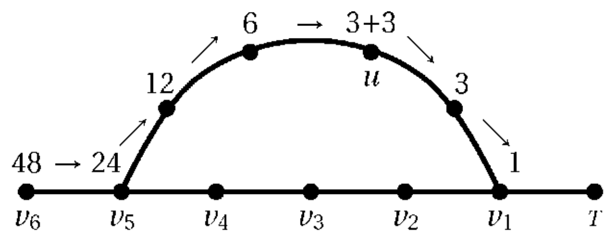


Figure 2.2: The result of sending all pebbles up the arch A.

u_1 and discard 3. Send 1 pebble to v_1, discard 1, and leave 1 at u_1. Then we cannot reach v_0 by sending all our pebbles up and across the arch A.

Let's instead send some pebbles across R and some pebbles up A, a strategy that utilizes both the shorter path R and the extra pebbles on the arch A:

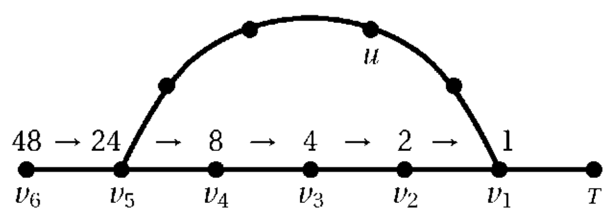


Figure 2.3: The first part of the strategy utilizing both R and A.

Send 24 pebbles to v_5 and discard 24. Send 8 pebbles to v_4 and discard 8, leaving 8 pebbles at v_5. Send 4 pebbles to v_3 and discard 4. Send 2 pebbles to v_2 and discard 2. Send 1 pebble to v_1 and discard 1.

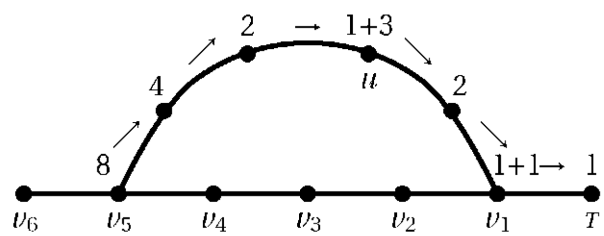


Figure 2.4: The second part of the strategy utilizing both R and A.

Of the 8 pebbles remaining at v_5, send 4 up to u_4 and discard 4. Send 2 pebbles to u_3 and discard 2. Send 1 pebble to u_2 and discard 1. We can now add this pebble to the 3 pebbles already sitting at u_2. Send 2 pebbles to u_1 and discard 2. Send 1 pebble to v_1 and discard 1. We can now add this pebble to the pebble already sitting at v_1. Send 1 pebble to v_0 and discard 1. Then yes, we CAN reach v_0 when $D(v_6) = 48$ and $D(u_2) = 3$!

Pebbling on this particular configuration can therefore utilize both the shorter path and the extra pebbles sitting on the arch. Optimizing the amount of pebbles sent up the arch versus the amount sent across the path will give us a better understanding of the value of the pebbling number of this graph.

Interesting questions arise when studying the graph described in Example 2.0.3, such as: What happens if we change the number of vertices on R or A?

What happens if we place a different number of pebbles at u_2 or v_7? What happens if more arches are added on top of our first arch? What generalizations can be made from our previous example?

2.1 First Trampoline Graph Definitions

The following definitions will play very important roles in this thesis and will be referenced often. Recall that $d(x, y)$ represents the distance between two vertices x and y.

Definition 2.1.1 *Let* $R = \{v_0, v_1, ..., v_{n-1} = v_N\}$ *and* $A = \{u_1, u_2, ..., u_n\}$. *The First Trampoline Graph has vertex set* $R \cup A$ *and edge set* $\{v_i v_{i+1} | i = 0, 1, ..., n-2 = N-1\} \cup \{u_i u_{i+1} | i = 1, 2, ..., n-1\} \cup \{v_j u_1\} \cup \{u_n v_i\}$. *Let* $u \in \{u_1, u_2, ..., u_n\}$ *be the vertex satisfying* $d(u, v_j) \leq d(v_i, u) \leq d(u, v_j) + 1$, *where* $d(v_i, u) < d(v_i, v_j) < d(u, v_j) + d(v_i, u)$.

While R denotes a subset of the vertex set of the first trampoline graph, we will also use R to indicate the subgraph induced by R; the distinction will be clear in context. Similarly, A will be used for both the vertex subset in Definition 2.1.1 and for the subgraph induced by this subset. Note that u is chosen to be either the vertex or one of the two vertices at maximum distance from R.

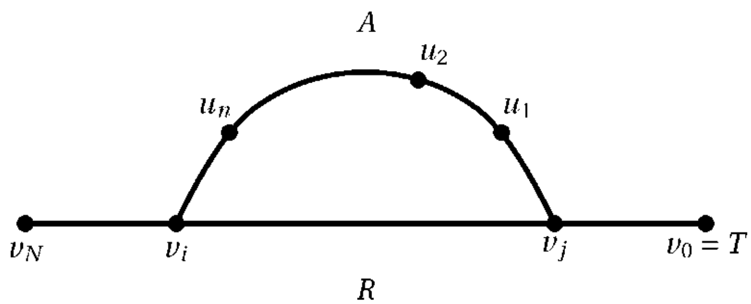


Figure 2.5: 1st-trampoline graph.

To shorten the notation, we will label $D = d(v_i, v_j)$, $\hat{d} = d(v_i, u)$, and $d = d(u, v_j)$. Then the inequalities in Definition 2.1.1 become $d \leq \hat{d} < D < d + \hat{d}$ and $\hat{d} \leq d + 1$.

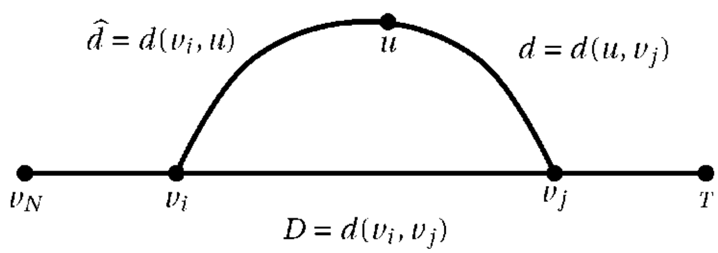


Figure 2.6: Distances considered in Definition 2.1.1.

The reason behind this distance requirement is that these trampoline graphs arise from considering a unique path of length $diam(G) = N$ from v_N to v_0. Hence, $D \neq d + \hat{d}$, otherwise there would be two different paths of length N between v_N and T. We also want to ensure that $d + \hat{d} \geq D$ because, if $d + \hat{d} < D$, there would exist a shorter path from v_N to T, and the diameter of the graph would no longer be N. This then forces $d + \hat{d} > D$.

By requiring that $d \leq \hat{d} \leq d + 1$, we guarantee that u is a farthest vertex from the path and therefore that we can place as many pebbles as possible on this vertex on the arch. A pebbling algorithm [1] is being developed for these particular configurations in a paper to appear by my advisor, Jorge Garcia.

Definition 2.1.2 *In a sequence of pebbling moves, a Joining Forces Point, or JF-Point, is a vertex that receives and combines pebbles from at least two paths.*

Let v_i and v_j be the vertices defined in Definition 2.1.1. Then v_i and v_j are the only two potential JF-points of the trampoline graph above.

Under the previous definitions, we have the following remark. This remark will be referred to again in Chapter 5 to show that the pebbling number of the first trampoline graph is greater than the pebbling number of the path P_n.

Remark 2.1.3 $2^N + 2^{(d+\hat{d})/2} - 2^{D-(d+\hat{d})/2} \geq 2^N + 2$.

Proof. Since $D > \hat{d} \geq (d+\hat{d})/2$, we have $D-(d+\hat{d})/2 > 0$, therefore $2^{D-(d+\hat{d})/2} \geq 2$. Since $(d + \hat{d})/2 > D - (d + \hat{d})/2$, then $2^{(d+\hat{d})/2} \geq 2^{D-(d+\hat{d})/2+1}$. Since $D \geq 3$, $2^{(d+\hat{d})/2} - 2^{D-(d+\hat{d})/2} \geq 2^{D-(d+\hat{d})/2} \geq 2$. (Note that Definition 2.1.1 would not hold if $D = 1$ or 2, so $D = 3$ is the smallest value such that $d \leq \widehat{d} < D < d + \widehat{d}$ is possible). Therefore $2^N + 2^{(d+\widehat{d})/2} - 2^{D-(d+\widehat{d})/2} \geq 2^N + 2$. □

The following remark helps to define the type of graph that qualifies as a first trampoline graph by comparing the length of A to the leg of the path R of length $N - i$.

Remark 2.1.4 *(Diameter Condition) When $d + \widehat{d} + D$ is an even cycle, $d + \widehat{d} \leq D + 2(N - i)$. When $d + \widehat{d} + D$ is an odd cycle, $d + \widehat{d} \leq D + 1 + 2(N - i)$.*

Proof. There exists a vertex w on the arch of distance k from v_i such that w is the furthest point from T on the arch.

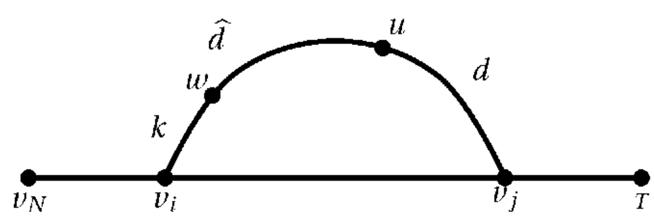


Figure 2.7: w on an even cycle.

When the cycle is even, we have $d(w, T) = i + k = d + \widehat{d} - k + j$, consequently $2k = d + \widehat{d} - i + j$, hence $k = (d + \widehat{d} - D)/2$. $k \leq N - i$, otherwise the diameter N of the graph would be violated. This implies that $(d + \widehat{d} - D)/2 \leq N - i$, therefore $d + \widehat{d} \leq D + 2(N - i)$.

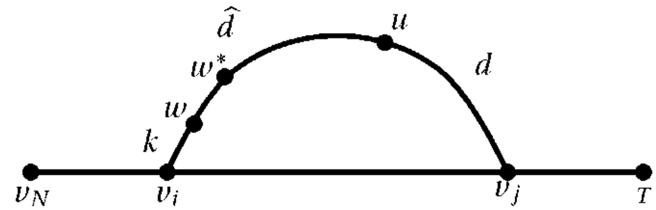


Figure 2.8: w and w^* on an odd cycle.

When the cycle is odd, $d(w,T) = i + k = d + \widehat{d} - k + j - 1 = d(w^*, T)$. This is because, while w is the farthest vertex from T, there exists another vertex w^* equally far from T. Thus $k = (d + \widehat{d} - D - 1)/2$, where $k \leq N - i$. This results in $d + \widehat{d} \leq D + 1 + 2(N - i)$. □

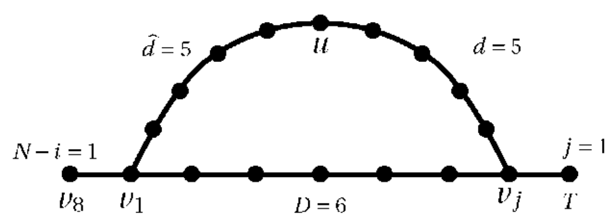


Figure 2.9: Example of a graph that does not meet assumption.

Definition 2.1.5 *A distribution on the first trampoline graph that has some pebbles at v_N and $u_i = u$ and zero pebbles at every other vertex satisfies the First Trampoline Property at u if:*

- *The distribution is solvable for $T = v_0$.*
- *Any sequence of pebbling moves that reaches T has exactly one JF - point.*
- *With fewer pebbles at either v_N or u, the distribution is unsolvable.*

Example 2.0.3 shown in Figure 4 is clearly an example of a distribution that satisfies the first trampoline property. The term *trampoline* comes from the idea that we "gain momentum" during our sequence of pebbling moves by sending some of our pebbles up the longer arch in order to utilize the extra pebbles sitting on the arch, then joining forces with the remaining pebbles that are sent across the path. In the following section, we will develop notation that will simplify the explanation of these pebbling moves.

2.2 Pebbling Moves Notation

The following notation will be utilized throughout this paper:

(a) $\mathbf{p}_{\rightarrow}(\mathbf{x})$ is the number of pebbles arriving at a vertex x.

(b) $\mathbf{p}(\mathbf{x})_{\rightarrow}$ is the number of pebbles leaving a vertex x.

(c) $\mathbf{s}^{-}_{@}(\mathbf{x})$ is the number of pebbles sitting at a vertex x before a sequence.

(d) $\mathbf{s}^{+}_{@}(\mathbf{x})$ is the number of pebbles sitting at a vertex x after a sequence.

To denote a sequence of pebbling moves that consists of sending z pebbles from v_x to v_y across R without using any other pebbles on R, we will utilize the following notation:

$$v_x \xrightarrow[R]{z} v_y$$

Example 2.2.1 *Consider the following distribution on* P_6

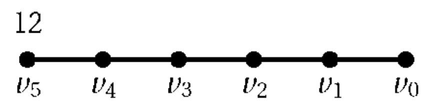


Figure 2.10: $D(v_5) = 12$ and $D(v_i) = 0$, $\forall\ v_i \in P_6 - \{v_5\}$.

and the following sequence of pebbling moves

$$v_5 \xrightarrow[R]{12} v_0 .$$

The sequence produces the following pebbling distribution.

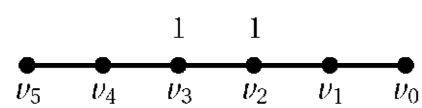


Figure 2.11: The result of sending 12 pebbles from v_5 across R.

Thus

$$v_5 \xrightarrow[R]{12} v_0 \quad \text{gives} \quad p_{\to}(v_0) = 0.$$

We will begin our exploration of graph pebbling by looking at particular configurations on the first trampoline graph and determining whether they satisfy the first trampoline property. After stating and proving various theorems regarding the possible values of pebbles at u and v_N, we will explore some examples related to these theorems. We will then do the same for second trampoline graphs. From here we will delve deeper into the idea of the pebbling number for the first trampoline graph.

CHAPTER 3

RESULTS ON FIRST TRAMPOLINE GRAPH

In this chapter, we will look more closely at the implications of the trampoline property on the 1st-trampoline graph. Specifically, we will explore how different amounts of pebbles placed on the vertices of the trampoline graph affect the strategies used to reach T.

To begin, we will revisit the definition of the first trampoline property. As before:

Definition 3.0.2 *A distribution on the first trampoline graph that has some pebbles at v_N and $u_i = u$ and zero pebbles at every other vertex satisfies the*

1st-trampoline property at u if:

- *The distribution is solvable for T.*
- *Any sequence of pebbling moves that reaches T has exactly 1 JF - point.*
- *With fewer pebbles at either v_N or u, the distribution is unsolvable.*

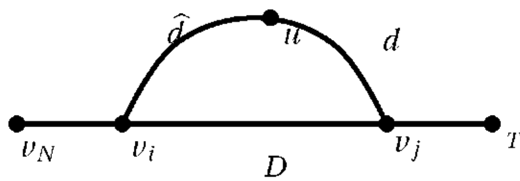


Figure 3.1: 1st-Trampoline Graph.

Theorem 3.0.3 *Under Definition 2.1.1, if we let* $D(u) = 2^d - 2^{D-\widehat{d}} + 1$ *and* $D(v) = 0$, $\forall\ v \in V(G) - \{u, v_N\}$, *then* $D(v_N) = 2^N - 2^{N-i+\widehat{d}}$ *is the smallest possible amount of pebbles at* V_N *such that the configuration is T-solvable.*

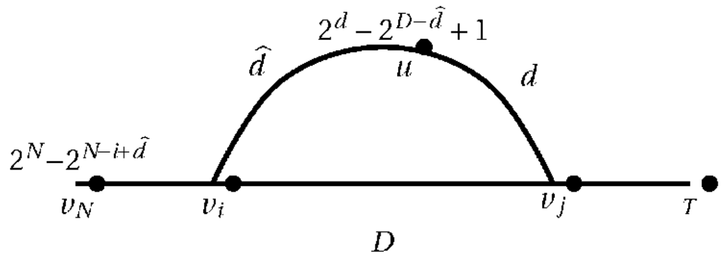


Figure 3.2: Configuration associated with Theorem 3.0.3.

Proof. In order to reach T, we will utilize only three methods: sending all pebbles along R, all along A, and splitting the pebbles between A and R. More explanation will be given later regarding why these three methods are the only necessary methods to utilize.

(i) First, let's try to send all $2^N - 2^{N-i+\hat{d}}$ pebbles from v_N along R.

$$v_N \xrightarrow[R]{2^N - 2^{N-i+\hat{d}}} v_i \quad \text{gives} \quad p_{\rightarrow}(v_i) = 2^i - 2^{\hat{d}}$$

$$v_i \xrightarrow[R]{2^i - 2^{\hat{d}}} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = 2^j - 1 \ (since\ \hat{d} < D)$$

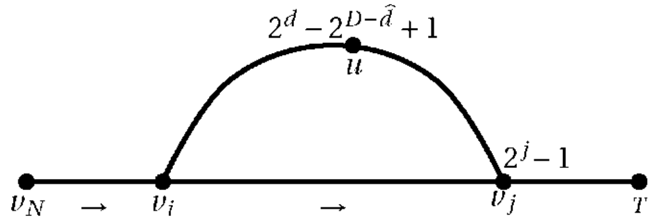


Figure 3.3: Strategy that sends all pebbles on v_N along R.

Also, since $D - \hat{d} > 0$, we have $2^{D-\hat{d}} > 1$. This implies $2^d - 2^{D-\hat{d}} + 1 < 2^d$. Hence no pebbles will arrive to v_j from u, thus

$$v_j \xrightarrow[R]{2^j - 1} T \quad \text{gives} \quad p_{\rightarrow}(T) = 0$$

Therefore we cannot reach T by sending all pebbles on v_N along R.

(ii) Next, let's try to send all $2^N - 2^{N-i+\hat{d}}$ pebbles from v_N along A.

$$v_N \xrightarrow[R]{2^N - 2^{N-i+\hat{d}}} v_i \quad \text{gives} \quad p_{\rightharpoonup}(v_i) = 2^i - 2^{\hat{d}}$$

$$v_i \xrightarrow[A]{2^i - 2^{\hat{d}}} u \quad \text{gives} \quad p_{\rightharpoonup}(u) = 2^{i-\hat{d}} - 1$$

Since $s^-_{@}(u) = 2^d - 2^{D-\hat{d}} + 1$, we have $s^+_{@}(u) = 2^d - 2^{D-\hat{d}} + 2^{i-\hat{d}}$.

Therefore since $D - \hat{d} - d < 0$, we conclude that

$$u \xrightarrow[A]{2^d + 2^{i-\hat{d}} - 2^{D-\hat{d}}} v_j \quad \text{gives} \quad p_{\rightharpoonup}(v_j) = 2^0 + 2^{i-\hat{d}-d} - 1 = 2^{i-\hat{d}-d}.$$

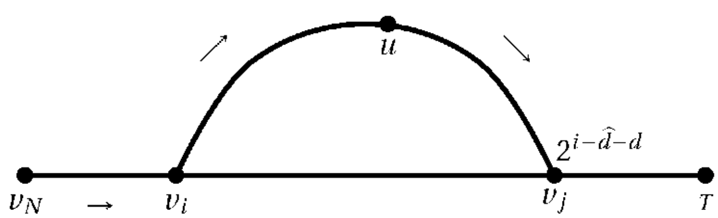


Figure 3.4: Strategy that sends all pebbles on v_N along A.

We know $\hat{d} + d > D$, therefore $2^{i-\hat{d}-d} < 2^{i-D} = 2^j$. This means $s^+_{@}(v_j) < 2^j$, which means that we cannot reach T by sending all pebbles along A.

(iii) Hence if the distribution is T-solvable, we must necessarily send some pebbles up the arch A and some along the path R. We will work backwards from T. In order for $p_{\rightharpoonup}(T) \geq 2^0 = 1$, we need at least 2^j pebbles at some point to be on v_j.

As shown in part (i), at least one pebble must arrive to v_j from u in order to reach T. It is not hard to see that, for every extra pebble that arrives to v_j from u, we need to send at least $2^{\hat{d}+d}$ pebbles from v_i to travel along the arch in order to reach v_j, but this is a waste of pebbles compared to the 2^D pebbles that can be sent directly along the path from v_i to reach v_j. Hence, when $u = 2^d - 2^{D-\hat{d}} + 1$, we can safely assume that only one pebble arrives to v_j from u. This means we at some point need 2^d pebbles at u to send them towards v_j, and the rest must come from v_i along the path. Therefore, at some point we need to make the following pebbling moves:

$$u \xrightarrow[A]{2^d} v_j \quad \text{and} \quad v_i \xrightarrow[R]{2^D(2^j-1)} v_j.$$

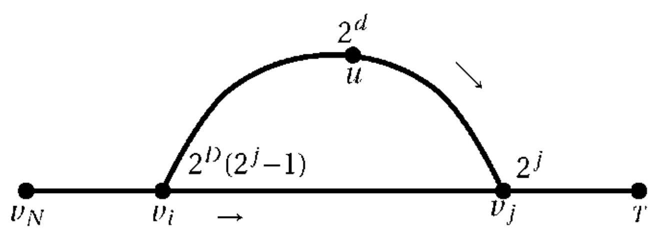


Figure 3.5: Pebbling moves to reach v_j.

Since our configuration has $2^d - 2^{D-\hat{d}} + 1$ pebbles on u initially, this forces u to receive $2^{D-\hat{d}} - 1$ pebbles. Hence we are forced to make the following pebbling moves:

$$v_i \xrightarrow[A]{(2^{D-\widehat{d}}-1)2^{\widehat{d}}} u \quad \text{and} \quad v_i \xrightarrow[R]{2^{D+j}-2^{D}} v_j$$

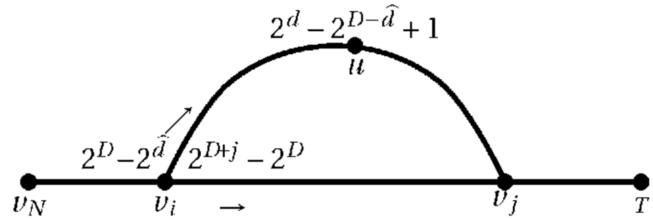


Figure 3.6: Pebbling moves from v_i.

Then we need at least $2^D - 2^{\widehat{d}} + 2^{D+j} - 2^D = 2^{D+j} - 2^{\widehat{d}}$ pebbles at v_i at some point. Clearly these pebbles must come from v_N, therefore we must make the following sequence of pebbling moves.

$$v_N \xrightarrow[R]{2^{N-i}(2^{D+j}-2^{\widehat{d}})} v_i$$

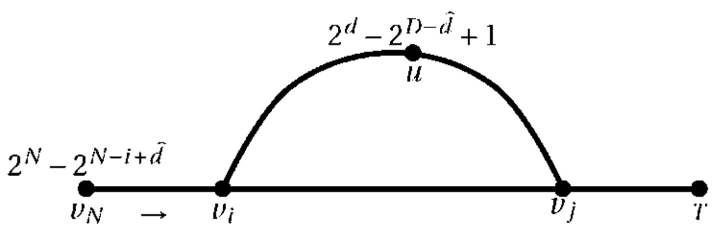


Figure 3.7: Initial configuration with pebbles on v_N and u.

Therefore the smallest amount of pebbles on v_N that we need in order to reach T is $2^{N-i}(2^{D+j} - 2^{\hat{d}}) = 2^N - 2^{N-i+\hat{d}}$. □

3.1 1st-Trampoline Property Significant Pebbling Cases

In this section we will distinguish between four natural ranges of pebbles on the first trampoline graph which result in theorems related to the first trampoline property. Notice that the previous proof only included three strategies:

(a) All along R

(b) All along A

(c) Some along R, some along A.

Since $d \le \hat{d} < D < d + \hat{d}$, it is of no benefit to send any of the pebbles at u back along the arch toward v_i, a path of length $\hat{d}$, because fewer (if any) pebbles would arrive at v_i than would at v_j.

If our configuration does not have enough pebbles on v_N to reach u, then there are not enough pebbles to reach v_j. Consider $w(x)$ to be the hypothetical amount of pebbles we "wish" to place at a vertex x.

What happens if we place different amounts of pebbles at u? First, note that it is only interesting to look at $w(u) < 2^{d+j}$, otherwise $w(u)$ could reach T without requiring extra pebbles from elsewhere.

If $0 < w(u) < 2^{\hat{d}}$, zero pebbles will reach v_i from u, and sending any pebbles towards v_i only sends the pebbles further from T. If $2^{\hat{d}} \leq w(u) < 2^{d+j}$, anywhere from 1 to 2^j pebbles reach v_j from u, and this is at most the amount that would reach v_i from u.

Since the path of length D is a shorter path than the arch of length $d + \hat{d}$, it makes sense to only send enough pebbles up our arch A so that after that sequence of pebbling moves, we have $p_{\rightarrow}(u) + s^{-}_{@}(u) = 2^d * c$ for $c \in \mathrm{N}, c < 2^j$, where $p_{\rightarrow}(u) < 2^D$. This will allow us to reach v_j using the pebbles from u without leaving any pebbles along the way (since $p_{\rightarrow}(v_j) = \lfloor \frac{p_{\rightarrow}(u) + s^{-}_{@}(u)}{2^d} \rfloor$).

But it turns out that when $c * 2^d < s^{-}_{@}(u) \leq 2^d * (c+1) - 2^{D-\hat{d}}$ for $c \in \mathrm{Z}$, $0 < c < 2^j$, the number of pebbles that we would need to send up A is so large that it is not worth the cost (since more pebbles are lost along the longer path A compared to the amount that would be lost along the shorter path R. As a result, there are four particular groups of $s^{-}_{@}(u) = x$ for $c \in \mathrm{N}, c < 2^j$ that are significant:

(1) $0 < x \leq 2^d - 2^{D-\hat{d}}$

(2) $c * 2^d - 2^{D-\hat{d}} < x \leq c * 2^d$

(3) $c * 2^d < x \le (c+1) * 2^d - 2^{D-\widehat{d}}$

(4) $2^{d+j} - 2^{D-\widehat{d}} < x < 2^{d+j}$

Theorem 3.1.1 *If we let $D(u) = x$ where $0 < x \le 2^d - 2^{D-\widehat{d}}$ and $D(v) = 0$, $\forall\ v \in V(G) - \{u, v_N\}$, then $D(v_N) = 2^N$ is the smallest amount of pebbles at v_N required to reach T.*

Proof. (a) First we note that the sequence of pebbling moves

$$v_N \xrightarrow[R]{2^N} T \quad \text{gives} \quad p_{\rightarrow}(T) = 1.$$

Hence we can reach T.

We will now show that if we place $2^N - 1$ pebbles at v_N and x pebbles at u, we cannot reach T.

Case 1: If we send all pebbles up A, then

$$v_N \xrightarrow[R]{2^N - 1} v_i \quad \text{gives} \quad p_{\rightarrow}(v_i) = 2^i - 1.$$

$$v_i \xrightarrow[A]{2^i - 1} u \quad \text{gives} \quad p_{\rightarrow}(u) = 2^{i-\widehat{d}} - 1.$$

Since $0 < x \le 2^d - 2^{D-\widehat{d}}$, after the previous sequences of pebbling moves we have $2^{i-\widehat{d}} - 1 < s^{+}_{@}(u) \le 2^{i-\widehat{d}} + 2^d - 2^{D-\widehat{d}} - 1$,

i.e. $2^{D+j-\widehat{d}} - 1 < s^{+}_{@}(u) \le 2^{D+j-\widehat{d}} + 2^d - 2^{D-\widehat{d}} - 1$.

Then $2^{D+j-\widehat{d}-d} - 1 < p_{\rightarrow}(v_j) \le 2^{D+j-\widehat{d}-d} + 1 - 2^{D-\widehat{d}-d} - 1$.

Therefore $p_{\rightarrow}(v_j) \leq 2^{D-\hat{d}-d}(2^j - 1)$. Since $2^{D-\hat{d}-d} < 1$, $p_{\rightarrow}(v_j) < 2^j$, therefore we cannot reach T.

Case 2: Send some pebbles up A and some along R.

We send pebbles along A only if we can combine these with the x pebbles on u to reach v_j once, and only if it is cheaper than sending enough pebbles along R to reach v_j once. However, in this case, in order to reach v_j by using A, we need to send $2^{\hat{d}}(2^d - x)$ pebbles from v_i along A to allow $2^d - x$ pebbles to arrive to u, thus $s^{-}_{@}(u) + s^{+}_{@}(u) = 2^d$ and we can reach v_j exactly once along A.

But $2^D \leq 2^{\hat{d}}(2^d - x)$ (since $x \leq 2^d - 2^{D-\hat{d}}$), i.e. it is cheaper to pebble v_j along R by sending 2^D pebbles across than by sending $2^{\hat{d}}(2^d - x)$ along A. Therefore this case reduces to Case 3.

Case 3: Sending all pebbles along R. Here

$$v_N \xrightarrow[R]{2^N - 1} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = 2^j - 1$$

$$u \xrightarrow[A]{x} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = 0.$$

Hence $p_{\rightarrow}(v_j) = 2^j - 1$ and we cannot reach T. This proves Theorem 3.1.1 □

Note that the configuration described in Theorem 3.1.1 does not satisfy the 1st-trampoline property. If we subtract one pebble from u (to allow $0 \leq$

$x \le 2^d - 2^{D-\hat{d}} - 1$), we can still reach T with the 2^N pebbles at v_N.

$$v_N \xrightarrow[R]{2^N} T \quad \text{gives} \quad p_{\rightharpoonup}(T) = 1.$$

Then $D(v_N) = 2^N$ is the smallest amount of pebbles needed at v_N to reach T, but we can still reach T with fewer pebbles at u. Hence the third requirement of the 1st-trampoline property is violated. In addition, this sequence in which all pebbles are sent along R results in zero JF-points on our graph, meaning the second requirement of the 1st-trampoline property is also violated.

Theorem 3.1.2 *If we let $D(u) = x$ where $c * 2^d < x \le (c+1) * 2^d - 2^{D-\hat{d}}$ for $c < 2^j, c \in \mathrm{N}$ and $D(v) = 0$, $\forall\ v \in V(G) - \{u, v_N\}$, then $D(v_N) = 2^{N-j}(2^j - c)$ is the smallest amount of pebbles at v_N required to reach T.*

Proof. (a) We observe that

$$v_N \xrightarrow[R]{2^{N-j}(2^j - c)} v_j \quad \text{gives} \quad p_{\rightharpoonup}(v_j) = 2^j - c.$$

$$u \xrightarrow[A]{x} v_j \quad \text{gives} \quad \lfloor x/2^d \rfloor.$$

Note that $\lfloor x/2^d \rfloor = c$ because $c * 2^d < x \le (c+1) * 2^d - 2^{D-\hat{d}}$. Then $s^+_{@}(v_j) = 2^j - c + c = 2^j$. Therefore we can reach T.

If we let $D(v_N) = 2^{N-j}(2^j - c) - 1$ and $D(u) = x$, where x still satisfies $c*2^d < x \leq (c+1)*2^d - 2^{D-\hat{d}}$ for $c < 2^j, c \in \mathrm{N}$, then we cannot reach T.

Case 1: Send all pebbles along R. We observe that

$$v_N \xrightarrow[R]{2^{N-j}(2^j - c) - 1} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = 2^j - c - 1.$$

$$u \xrightarrow[A]{x} v_j \quad \text{gives} \quad c.$$

then $s^+_{@}(v_j) = 2^j - 1$, which is not enough to reach T.

Case 2: Send some pebbles up A and some along R.

We send enough pebbles from u to v_j to facilitate the maximum amount reaching v_j. This is $\lfloor x/2^d \rfloor$, which from the hypothesis is equal to c. Then we send $c * 2^d$ from u to v_j. We have $x - c * 2^d$ pebbles left on u. Now we send pebbles along A only if we can combine these with the pebbles at u to pebble v_j once, and only if it is cheaper than sending enough pebbles along R to reach v_j. In this case, in order to pebble v_j along A, we need to send $2^{\hat{d}}(2^d - x + 2^d * c)$ pebbles for $2^d - x + 2^d * c$ pebbles to arrive at u, and once combined with the $x - 2^d * c$ pebbles at u, we obtain 2^d, which is enough to reach v_j once. However, $2^D \leq 2^{\hat{d}}(2^d - x + 2^d * c)$ since $x \leq (c+1) * 2^d - 2^{D-\hat{d}}$, which means it is cheaper to pebble v_j directly along R by sending 2^D pebbles toward v_j. Therefore this case reduces to Case 1.

Case 3: Send all pebbles up A.

$$v_N \xrightarrow[R]{2^{N-j}(2^j-c)-1} v_i \quad \text{gives} \quad p_{\rightarrow}(v_i) = 2^D(2^j-c)-1$$
$$v_i \xrightarrow[A]{2^D(2^j-c)-1} u \quad \text{gives} \quad p_{\rightarrow}(u) = 2^{D-\hat{d}}(2^j-c)-1$$

Note that $s^-_{@}(u) = x$, hence $s^+_{@}(x) = 2^{D-\hat{d}}(2^j-c)-1+x$. Observe that

$$u \xrightarrow[A]{2^{D-\hat{d}}(2^j-c)-1+x} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = \lfloor \frac{2^{D-\hat{d}}(2^j-c)-1+x}{2^d} \rfloor$$

Where

$$\begin{aligned}
\frac{2^{D-\hat{d}}(2^j-c)-1+x}{2^d} &\leq \frac{2^{D-\hat{d}}(2^j-c)-1+(c+1)2^d-2^{D-\hat{d}}}{2^d} \\
&= 2^{D-\hat{d}-d}(2^j-c)-1/2^d+c+1-2^{D-\hat{d}-d} \\
&= 2^{D-\hat{d}-d+j}-c*2^{D-\hat{d}-d}+c+1-1/2^d-2^{D-\hat{d}-d} \\
&= c(1-2^{D-\hat{d}-d})+2^{D-\hat{d}-d+j}+1-1/2^d-2^{D-\hat{d}-d} \\
&\leq (2^j-1)(1-2^{D-\hat{d}-d}+2^{D-\hat{d}-d+j}+1-1/2^d-2^{D-\hat{d}-d} \\
&= 2^j-1-2^{D-\hat{d}-d+j}+2^{D-\hat{d}-d}+2^{D-\hat{d}-d+j}+1-1/2^d-2^{D-\hat{d}-d} \\
&= 2^j-1/2^d.
\end{aligned}$$

Then $\lfloor \frac{2^{D-\hat{d}}(2^j-c)-1+x}{2^d} \rfloor \leq \lfloor 2^j-1/2^d \rfloor = 2^j-1$. Hence $s^+_{@}(v_j) \leq 2^j-1$. We cannot reach T, therefore this proves Theorem 3.1.2. □

We note that the configuration described in Theorem 3.1.2 does not satisfy the 1st-trampoline property. If we remove one of the pebbles from u, we now have $D(u) = x$, where $c * 2^d \le x \le (c+1) * 2^d - 2^{D-\hat{d}} - 1$. Then

$$v_N \xrightarrow[R]{2^{N-j}(2^j - c)} v_j \quad \text{gives} \quad p_{\rightharpoonup}(v_j) = 2^j - c$$

$$u \xrightarrow[A]{x} v_j \quad \text{still gives} \quad c.$$

Hence $p_{\rightharpoonup}(v_j) = 2^j - c + c = 2^j$. Therefore we can still reach T with our new distribution, which means the third requirement of the 1st- trampoline property is violated. Unlike the range of x in Theorem 3.1.1, this range of x will have 1 JF-point, hence the second requirement of the 1st-trampoline property is met.

If $c{*}2^d < x \le (c{+}1){*}2^d - 2^{D-\hat{d}}$ for $c{<}2^j, c{\in}\mathrm{N}$ then $2^{N-j}(2^j - c)$ is the smallest amount of pebbles at v_N required to reach T, however this distribution does not satisfy the third requirement of the 1st-trampoline property.

Theorem 3.1.3 *If we let $D(u) = x$ where $2^{d+j} - 2^{D-\hat{d}} < x < 2^{d+j}$ and $D(v) = 0$, $\forall\ v \in V(G) - \{u, v_N\}$, then $D(v_N) = 2^{N-i}[2^{\hat{d}}(2^{d+j} - x)]$ is the smallest amount of pebbles required at v_N to reach T.*

Proof. (a) Observe that

$$v_N \xrightarrow[R]{2^{N-i}[2^{\hat{d}}(2^{d+j} - x)]} v_i \quad \text{gives} \quad p_{\rightharpoonup}(v_i) = 2^{\hat{d}}(2^{d+j} - x).$$

$$v_i \xrightarrow[A]{2^{\hat{d}}(2^{d+j} - x)} u \quad \text{gives} \quad p_{\rightharpoonup}(u) = 2^{d+j} - x.$$

Note that $s^-_{@}(u) = x$, hence $s^+_{@}(u) = 2^{d+j}$. Now observe that

$$u \xrightarrow[A]{2^{d+j}} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = 2^j.$$
$$v_j \xrightarrow[R]{2^j} T \quad \text{gives} \quad p_{\rightarrow}(T) = 1.$$

Therefore we can reach T with this amount of pebbles at v_N. But if we subtract 1 of the pebbles from v_N, we would have $D(v_N) = 2^{N-i}[2^{\hat{d}}(2^{d+j} - x)] - 1$ pebbles at v_N. With $D(v_N) = 2^{N-i}[2^{\hat{d}}(2^{d+j} - x)] - 1$ and $D(u) = x$, where $2^{d+j} - 2^{D-\hat{d}} < x < 2^{d+j}$, we cannot reach T.

$$v_N \xrightarrow[R]{2^{N-i}[2^{\hat{d}}(2^{d+j} - x)] - 1} v_i \quad \text{gives} \quad p_{\rightarrow}(v_i) = 2^{\hat{d}}(2^{d+j} - x) - 1$$

Hence $2^{\hat{d}}(2^{d+j} - 2^{d+j}) - 1 < p_{\rightarrow}(v_i) < 2^{\hat{d}}(2^{d+j} - 2^{d+j} + 2^{D-\hat{d}}) - 1$. In other words, $0 \leq p_{\rightarrow}(v_i) < 2^D - 1$.

This means that the only helpful way of utilizing these pebbles is to send all of them up A because there are not enough to reach v_j by sending any along R.

$$v_i \xrightarrow[A]{2^{\hat{d}}(2^{d+j} - x) - 1} u \quad \text{gives} \quad p_{\rightarrow}(u) = 2^{d+j} - x - 1$$
$$u \xrightarrow[A]{2^{d+j} - 1} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = 2^j - 1$$

Hence, we cannot reach T when we place $D(v_N) = 2^{N-i}[2^{\hat{d}}(2^{d+j} - x)] - 1$ pebbles at v_N. Then $D(v_N) = 2^{N-i}[2^{\hat{d}}(2^{d+j} - x)]$ is the smallest amount of pebbles needed to reach T when $2^{d+j} - 2^{D-\hat{d}} < x < 2^{d+j}$. This proves Theorem 3.1.3.□

Unfortunately, sending all pebbles up A means that this distribution of pebbles will have zero JF-points, therefore the distribution described in Theorem 3.1.3 does not satisfy the second requirement of the 1st-trampoline property.

Therefore if $D(u) = x$, where $2^{d+j} - 2^{D-\hat{d}} < x < 2^{d+j}$, then $D(v_N) = 2^{N-i}[2^{\hat{d}}(2^{d+j} - x)]$ is the smallest amount of pebbles required at v_N to reach T, however this distribution does not satisfy the second requirement of the 1st-trampoline property.

Theorem 3.1.4 *If we let* $D(u) = x$ *where* $c * 2^d - 2^{D-\hat{d}} < x \leq c * 2^d$ *and* $D(v) = 0$, $\forall\ v \in V(G) - \{u, v_N\}$, *then specifying* $D(v_N) = 2^{N-i}[2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x)]$ *results in* D *satisfying the 1st-trampoline property.*

Proof. (1) The distribution is solvable for T.

Let $c * 2^d - 2^{D-\hat{d}} < x < c * 2^d$.Observe that

$$\begin{aligned}
v_N \xrightarrow[R]{2^{N-i}[2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x)]} v_i \quad &\text{gives} \quad p_{\rightarrow}(v_i) = 2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x) \\
v_i \xrightarrow[R]{2^D(2^j - c)} v_j \quad &\text{gives} \quad p_{\rightarrow}(v_j) = 2^j - c \\
v_i \xrightarrow[A]{2^{\hat{d}}(c * 2^d - x)} u \quad &\text{gives} \quad p_{\rightarrow}(u) = c * 2^d - x \\
u \xrightarrow[A]{c * 2^d} v_j \quad &\text{gives} \quad p_{\rightarrow}(v_j) = c.
\end{aligned}$$

Hence we have $c + 2^j - c = 2^j$ pebbles at v_j, enough to reach T.

Let $x = c * 2^d$. Then $2^{N-i}[2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x)] = 2^{N-i}[2^D(2^j - c)]$.

$$v_N \xrightarrow[R]{2^{N-i}[2^D(2^j-c)]} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = 2^j - c$$
$$u \xrightarrow[A]{c*2^d} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = c$$

Therefore, as before, total $p_{\rightarrow}(v_j) = c+2^j-c = 2^j$. Then 1 pebble will arrive at T in both cases, therefore the distribution is solvable for T when $c * 2^d - 2^{D-\hat{d}} < x \leq c * 2^d$.

(2) Any sequence of pebbling moves that is T-solvable has exactly 1 JF point.

i) v_j is a JF point. Suppose not.

Suppose all pebbles travel along R. We have already proved that this is not possible because we would need at least $D(v_N) = 2^N$ pebbles at v_N to accomplish this, and $2^{N-i}[2^D(2^j-c)+2^{\hat{d}}(c*2^d-x)] < 2^N$ when $c*2^d-2^{D-\hat{d}} < x < c * 2^d$.

Suppose all pebbles travel along A. Then we would need at least $D(v_N) = 2^{N-i+\hat{d}}(2^{d+j} - x)$, but $2^{N-i+\hat{d}}(2^{d+j} - x) > 2^{N-i}[2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x)]$ (since $2^D(2^j - c) + 2^{\hat{d}}(c*2^d - x) < 2^{\hat{d}}(2^{d+j} - x)$). Therefore we must send some pebbles along R and some pebbles along A. Then at least 1 pebble arrives at

v_j from A and 1 from R, hence v_j is a JF-point.

ii) Suppose v_i is another JF-point. Clearly with $D(u) = x$ where $c * 2^d - 2^{D-\hat{d}} < x \leq c * 2^d$, at most c pebbles would arrive to v_i, and that would only occur when $D(u) = 2^d$. At this point, sending pebbles up A is worse than sending all pebbles along R, hence we would need to combine the c pebbles with the $2^{N-i}\lfloor 2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x)\rfloor$ pebbles at v_N, such that

$$\begin{aligned} P(v_i)_{\rightarrow} &= 2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x) + c \\ &< 2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - c * 2^d + 2^{D-\hat{d}}) + c = 2^i - c * 2^D + 2^D + c, \end{aligned}$$

thus $P(v_i)_{\rightarrow} < 2^i$ when $c*2^d - 2^{D-\hat{d}} < x \leq c*2^d$, therefore v_i could not be a JF point.

(3) With fewer pebbles, the distribution is unsolvable.

i) Let $s^{-}_{@}(v_N) = 2^{N-i}\lfloor 2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x)\rfloor - 1$ and $c * 2^d - 2^{D-\hat{d}} < x \leq c * 2^d$.

Case 1: Send all pebbles along R to v_j.

$$v_N \xrightarrow[R]{2^{N-i}\lfloor 2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x)\rfloor - 1} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = 2^j - c + 2^{\hat{d}-D}(c*2^d - x) - 1$$

$$u \xrightarrow[A]{x} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = \lfloor x/2^d \rfloor$$

Then total $p_{\rightarrow}(v_j) < 2^j - c + c(2^{\hat{d}+d-D}) - (c*2^d - 2^{D-\hat{d}})2^{\hat{d}-D} - 1 + \lfloor (c*2^d - 2^{D-\hat{d}})/2^d \rfloor$

$= 2^j - c + c(2^{\hat{d}+d-D}) - c(2^{\hat{d}+d-D}) + 1 - 1 + c - 1 = 2^j - c + c - 1 = 2^j - 1.$

Therefore we cannot reach T.

Case 2: Send all pebbles up A to u.

$$v_N \xrightarrow[R]{2^{N-i}[2^D(2^j-c)+2^{\hat{d}}(c*2^d-x)]-1} v_i \quad \text{gives} \quad p_{\rightarrow}(v_i)=2^D(2^j-c)+2^{\hat{d}}(c*2^d-x)-1$$

$$v_i \xrightarrow[A]{2^D(2^j-c)+2^{\hat{d}}(c*2^d-x)-1} u \quad \text{gives} \quad p_{\rightarrow}(u)=2^{D-\hat{d}}(2^j-c)+c*2^d-x-1$$

$$u \xrightarrow[A]{2^{D-\hat{d}}(2^j-c)+c*2^d-1} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j)=2^{D-\hat{d}-d}(2^j-c)+c-1$$

Where $2^{D-\hat{d}+d}<1$, hence $2^{D-\hat{d}-d}(2^j-c)<2^j-c$.

Therefore we cannot reach T.

Case 3: Send some pebbles along R, some along A.

$$(a)\, v_i \xrightarrow[A]{2^{\hat{d}}(c*2^d-x)-1} u \quad \text{gives} \quad P(u)_{\rightarrow}=c*2^d-x-1$$

$$u \xrightarrow[A]{c*2^d-1} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j)=c-1$$

$$v_i \xrightarrow[R]{2^D(2^j-c)} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j)=2^j-c.$$

Therefore we cannot reach T.

$$(b)\, v_i \xrightarrow[A]{2^{\hat{d}}(c*2^d-x)} u \quad \text{gives} \quad P(u)_{\rightarrow}=c*2^d-x$$

$$u \xrightarrow[A]{c*2^d} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j)=c$$

$$v_i \xrightarrow[R]{2^D(2^j-c)-1} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j)=2^j-c-1.$$

Therefore we cannot reach T.

ii) Let $s^-_{@}(v_N)=2^{N-i}[2^D(2^j-c)+2^{\hat{d}}(c*2^d-(x+1))]$ and $c*2^d-2^{D-\hat{d}}\leq x\leq$

$c * 2^d - 1$.

Case 1: Send all pebbles along R To v_j.

$$v_N \xrightarrow[R]{2^{N-i}[2^D(2^j-c)+2^{\hat{d}}(c*2^d-x-1)]} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = 2^j - c + 2^{\hat{d}-D}(c*2^d-x-1)$$

$$u \xrightarrow[A]{x} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = \lfloor x/2^d \rfloor$$

Then total $p_{\rightarrow}(v_j) = 2^j - c + c * 2^{\hat{d}+d-D} - \lfloor (x+1) * 2^{\hat{d}-D} \rfloor + \lfloor x/2^d \rfloor$, i.e.

$p_{\rightarrow}(v_j) \leq 2^j - c + c(2^{\hat{d}+d-D}) - c * 2^{\hat{d}+d-D} + 1 - 1 + \lfloor (c * 2^d - 2^{D-\hat{d}})/2^d \rfloor$

$= 2^j - c + c - 1 = 2^j - 1$.

Therefore we cannot reach T.

Case 2: Send all pebbles up A to u.

$$v_N \xrightarrow[R]{2^{N-i}[2^D(2^j-c)+2^{\hat{d}}(c*2^d-x-1)]} v_i \quad \text{gives} \quad p_{\rightarrow}(v_i) = 2^D(2^j-c) + 2^{\hat{d}}(c*2^d-x-1)$$

$$v_i \xrightarrow[A]{2^D(2^j-c)+2^{\hat{d}}(c*2^d-x-1)} u \quad \text{gives} \quad p_{\rightarrow}(u) = 2^{D-\hat{d}}(2^j-c) + c*2^d - x - 1$$

$$u \xrightarrow[A]{2^{D-\hat{d}}(2^j-c)+c*2^d-1} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = \lfloor 2^{D-\hat{d}-d}(2^j-c) \rfloor + c - 1$$

Where $\lfloor 2^{D-\hat{d}-d}(2^j - c) \rfloor < 2^j - c$, hence $p_{\rightarrow}(v_j) < 2^j - c + c - 1 = 2^j - 1$,

Therefore we cannot reach T.

Case 3: Send some pebbles along R, some along A.

$$(a)\, v_i \xrightarrow[A]{2^{\hat{d}}(c*2^d-x-1)} u \quad \text{gives} \quad P(u)_{\rightarrow} = c*2^d - x - 1$$

$$u \xrightarrow[A]{c*2^d-1} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = c - 1$$

$$v_i \xrightarrow[R]{2^D(2^j - c)} v_j \quad \text{gives} \quad p_{\rightarrow}(u) = 2^j - c.$$

Therefore we cannot reach T.

$$(b)\, v_i \xrightarrow[A]{2^{\hat{d}}(c * 2^d - x)} u \quad \text{gives} \quad P(u)_{\rightarrow} = c * 2^d - x$$

$$u \xrightarrow[A]{c * 2^d} v_j \quad \text{gives} \quad p_{\rightarrow}(v_j) = c$$

$$v_i \xrightarrow[R]{2^D(2^j - c) - 2^{\hat{d}}} v_j \quad \text{gives} \quad p_{\rightarrow}(u) = 2^j - c - 1.$$

Therefore we cannot reach T.

Thus if we let $D(u) = x$ where $c * 2^d - 2^{D-\hat{d}} < x \leq c * 2^d$ and $D(v) = 0$, $\forall\ v \in V(G) - \{u, v_N\}$, then $D(v_N) = 2^{N-i}[2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - x)]$ satisfies the 1st-trampoline property. $\square$

3.2 1st-Trampoline Property Examples

Recall that there were four significant cases for the amount of pebbles at u.

These are:

For $c < 2^j, c \in N$

(1) $0 < x \leq 2^d - 2^{D-\hat{d}}$

(2) $c * 2^d - 2^{D-\hat{d}} < x \leq c * 2^d$

(3) $c * 2^d < x \leq (c+1) * 2^d - 2^{D-\widehat{d}}$

(4) $2^{d+j} - 2^{D-\widehat{d}} < x < 2^{d+j}$

Now consider $d = 3, \widehat{d} = 4, D = 5, j = 2, N - i = 2$. Then $c < 4$. This means that there are 4 significant cases/ranges for x:

(1) $0 < x \leq 6$

(2) $6 < x \leq 8$, $14 < x \leq 16$, $22 < x \leq 24$

(3) $8 < x \leq 14$, $16 < x \leq 22$, $24 < x \leq 30$

(4) $30 < x < 32$

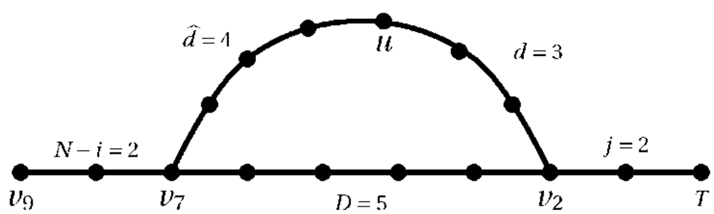


Figure 3.8: Example of the 1st- trampoline graph.

Example 3.2.1 *If we let $D(u) = x$, where $0 < x \leq 6$, fewer pebbles will be lost if we send all pebbles from v_9 straight along R to reach T. Hence, we will always place 512 pebbles at v_9 in order to reach T.*

Let's place $x = 3$ pebbles at u. Then if we want to send some pebbles up A and some along R, we would need at least 704 pebbles at v_9 to reach T.

$$v_9 \xrightarrow[R]{704} v_7 \quad \text{gives} \quad p_{\rightarrow}(v_7) = 176$$

$$v_7 \xrightarrow[R]{96} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 3$$

$$v_7 \xrightarrow[A]{80} u_3 \quad \text{gives} \quad p_{\rightarrow}(u_3) = 5$$

$$u_3 \xrightarrow[A]{8} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 1$$

$$v_2 \xrightarrow[R]{4} T \quad \text{gives} \quad p_{\rightarrow}(T) = 1$$

If we instead send all of our pebbles up A, we would need a total of 1856 pebbles at v_9.

But with only $2^9 = 512$ pebbles, we can reach T directly by sending all 512 pebbles along R. This means that this distribution of x does not satisfy the 1st-trampoline property because there are zero JF-points with this configuration.

Example 3.2.2 *If we let $D(u) = x$, where $6 < x \leq 8$, $14 < x \leq 16$, and $22 < x \leq 24$, fewer pebbles will be lost if we send some pebbles from v_9 along A and some along R to reach T. We will always satisfy the 1st-trampoline property if we let $D(v_9) = 2^7(2^2 - c) + 2^6(c * 2^3 - x)$.*

(1) Place 7 pebbles at u. This occurs when $c = 1$. Then if we want to send some pebbles up A and some along R, we would need 448 pebbles at v_9.

$$v_9 \xrightarrow[R]{448} v_7 \quad \text{gives} \quad p_{\rightarrow}(v_7) = 112$$
$$v_7 \xrightarrow[R]{96} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 3$$
$$v_7 \xrightarrow[A]{16} u_3 \quad \text{gives} \quad p_{\rightarrow}(u_3) = 1$$
$$u_3 \xrightarrow[A]{8} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 1$$
$$v_2 \xrightarrow[R]{4} T \quad \text{gives} \quad p_{\rightarrow}(T) = 1$$

If we instead send all of our pebbles up A, we would need a total of 1600 pebbles at v_9.

Sending all pebbles along R would require 512 pebbles at v_9, therefore it is most cost efficient to send some pebbles up A and some along R.

(2) Let u have 15 pebbles. This occurs when $c = 2$. Then if we want to send some pebbles up A and some along R, we would require that v_9 have 320 pebbles.

$$v_9 \xrightarrow[R]{320} v_7 \quad \text{gives} \quad p_{\rightarrow}(v_7) = 80$$
$$v_7 \xrightarrow[R]{64} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 2$$
$$v_7 \xrightarrow[A]{16} u_3 \quad \text{gives} \quad p_{\rightarrow}(u_3) = 1$$
$$u_3 \xrightarrow[A]{16} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 2$$

$$v_2 \xrightarrow[R]{4} T \quad \text{gives} \quad p_{\to}(T) = 1$$

If we instead send all of our pebbles up A, we would need a total of 1088 pebbles at v_9.

Sending all pebbles along R would again require 512 pebbles at v_9, therefore it is most cost efficient to send some pebbles up A and some along R.

(3) Let's place $x = 24$ pebbles at u. This occurs when $c = 3$. Then if we want to send all of our pebbles along R, we would need 128 pebbles at v_9.

$$v_9 \xrightarrow[R]{128} v_2 \quad \text{gives} \quad p_{\to}(v_2) = 1$$

$$u_3 \xrightarrow[A]{24} v_2 \quad \text{gives} \quad p_{\to}(v_2) = 3$$

$$v_2 \xrightarrow[R]{4} T \quad \text{gives} \quad p_{\to}(T) = 1$$

If we instead send all of our pebbles up A, we would need a total of 512 pebbles at v_9.

Sending some pebbles along R and some up A would not be very cost efficient because we would need to send up at least 128 pebbles in order for 4 pebbles to arrive at v_2, which would again require 512 pebbles at v_9, therefore it is most cost efficient to send all pebbles along R.

These particular distributions of x therefore satisfy the 1st-trampoline property.

Example 3.2.3 *If we let* $D(u) = x$*, where* $8 < x \leq 14$, $16 < x \leq 22$*, and* $24 < x \leq 30$*, it will always be most cost efficient to send all pebbles from* v_9 *straight along* R *to reach* T.

(1) Let's place 11 pebbles at u ($c = 1$) and 384 pebbles at v_9. Then if we want to send some pebbles up A and some along R,

$$v_9 \xrightarrow[R]{384} v_7 \quad \text{gives} \quad p_{\rightarrow}(v_7) = 96$$

$$v_7 \xrightarrow[R]{16} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 0$$

$$v_7 \xrightarrow[A]{80} u_3 \quad \text{gives} \quad p_{\rightarrow}(u_3) = 5$$

$$u_3 \xrightarrow[R]{16} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 2$$

This means we cannot reach T by sending this amount of pebbles both up A and along R. Sending any fewer pebbles up A would result in only 1 pebble arriving to v_2 from u_3, which is the same number that would arrive with the 11 pebbles already sitting at u_3. Therefore

$$v_9 \xrightarrow[R]{384} v_7 \quad \text{gives} \quad p_{\rightarrow}(v_7) = 96$$

$$v_7 \xrightarrow[R]{96} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 3$$

$$u_3 \xrightarrow[A]{11} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 1$$

$$v_2 \xrightarrow[R]{4} T \quad \text{gives} \quad p_{\rightarrow}(T) = 1$$

Again, sending all pebbles up A would require 1344 pebbles at v_9, thus it is most cost efficient to send all pebbles along R.

(2) Let's place 20 pebbles at u ($c = 2$) and 256 pebbles at v_9. This means

$$v_9 \xrightarrow[R]{256} v_7 \quad \text{gives} \quad p_{\rightarrow}(v_7) = 64$$

Then if we want to send some pebbles up A and some along R, we would need to send at least 32 across R in order to reach v_2, but this means only 32 pebbles are sent up A, where

$$v_7 \xrightarrow[A]{32} u_3 \quad \text{gives} \quad p_{\rightarrow}(u_3) = 2$$

$$u_3 \xrightarrow[A]{22} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 2$$

Which means that $s^{+}_{@}(v_2) = 3$, which is not enough to reach T.

Sending all pebbles across A,

$$v_7 \xrightarrow[A]{64} u_3 \quad \text{gives} \quad p_{\rightarrow}(u_3) = 4$$

$$u_3 \xrightarrow[A]{24} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 3$$

Which is again not enough to reach T.

Sending all pebbles from v_7 across R and bringing the pebbles towards v_j from u,

$$v_7 \xrightarrow[R]{64} v_2 \quad \text{gives} \quad p_{\rightarrow}(u_3) = 2$$

$$u_3 \xrightarrow[A]{20} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 2$$

Therefore, with 256 pebbles on v_9 and 20 pebbles on u_3, it is only possible to reach T by sending all pebbles from v_9 across R.

(3) Let's place 25 pebbles at u ($c = 3$) and 128 pebbles at v_9.

If we want to send some pebbles up A and some along R, we would need to send at least 112 pebbles up A for 4 pebbles to reach v_2. In this particular case, no pebbles would need to be sent along R, therefore a total of 448 pebbles would be required at v_9. But if we send all pebbles along R,

$$v_9 \xrightarrow[R]{128} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 1$$

$$u_3 \xrightarrow[A]{25} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 3$$

$$v_2 \xrightarrow[R]{4} T \quad \text{gives} \quad p_{\rightarrow}(T) = 1.$$

These ranges of pebbles at x and v_9 have 1 JF-point and are T-solvable, but with only 24 pebbles at u_3 we could still reach T, therefore this distribution does not satisfy the 1st-trampoline property.

Example 3.2.4 *If we let $D(u) = x$, where $30 < x < 32$, fewest pebbles are lost when all pebbles from v_9 are sent up the arch A to reach T.*

Let's place 31 pebbles at u and 64 pebbles at v_9. Sending all 128 pebbles along the path would result in

$$v_9 \xrightarrow[R]{64} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 0$$

$$u_3 \xrightarrow[A]{31} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 3.$$

This means we cannot reach T by sending the pebbles at v_9 along the path R.

Sending some pebbles along the path will not work because at least 16 pebbles must be sent up the arch in order to make a difference in the pebbles that reach v_2 from the arch, which leaves us with the case when all pebbles are sent up the arch.

$$v_9 \xrightarrow[R]{64} v_7 \quad \text{gives} \quad p_{\rightarrow}(v_7) = 16$$

$$v_7 \xrightarrow[A]{16} u_3 \quad \text{gives} \quad p_{\rightarrow}(u_3) = 1$$

$$u_3 \xrightarrow[A]{32} v_2 \quad \text{gives} \quad p_{\rightarrow}(v_2) = 4$$

Therefore we can reach T by sending all of our pebbles up the arch, and this is the most cost efficient pebbling method.

This range of pebbles at x has zero JF-points, therefore this distribution does not satisfy the 1st-trampoline property.

CHAPTER 4

Results On Larger Trampoline Graphs

The implications of the trampoline property on the 1st-trampoline graph are similar to the implications of the trampoline property on the 2nd-trampoline graph. Example of the 2nd-trampoline graph:

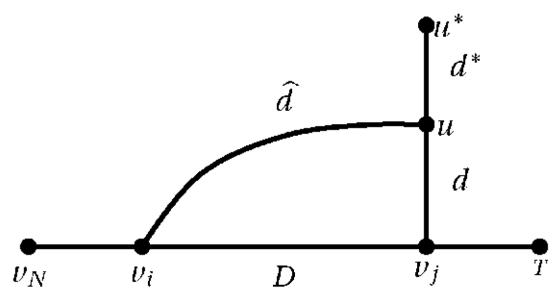


Figure 4.1: 2nd-Trampoline Graph.

Definition 4.0.5 *Let $R = \{v_0, v_1, ..., v_N\}$, $A = \{u_1, u_2, ..., u_n\}$, and $B = \{u_1^*, u_2^*, ..., u_m^*\}$. The Second Trampoline Graph has vertex set $R \cup A \cup B$ and edge set $\{v_i v_{i+1} | i = 0, 1, ..., N-1\} \cup \{u_i u_{i+1} | i = 1, 2, ..., n-1\} \cup \{u_i^* u_{i+1}^* | i = 1, 2, ..., m-1\} \cup \{v_j u_1\} \cup \{u_n v_i\} \cup \{u u_1^*\}$. Let $u \in \{u_1, u_2, ..., u_n\}$ be the vertex satisfying $d(u, v_j) \leq d(v_i, u) \leq d(u, v_j) + 1$, where $d(v_i, u) < d(v_i, v_j) < d(u, v_j) + d(v_i, u)$.*

To shorten the notation, we will label $D = d(v_i, v_j)$, $\widehat{d} = d(v_i, u)$, and $d = d(u, v_j)$, as before. Then the inequalities in Definition 4.0.5 become $d \leq \widehat{d} < D < d + \widehat{d}$ and $\widehat{d} \leq d + 1$.

We will also label $d^* = d(u, u^*)$. To ensure that the diameter of the graph remains N, we must place restrictions on d^*. By setting $d^* \leq N - j - d$, we ensure that $d(u^*, v_0) \leq N$. Similarly, setting $d^* \leq i - \widehat{d}$ guarantees that $d(u^*, v_N) \leq N$ Combining and simplifying these inequalities, $d^* \leq \frac{N-1}{2}$.

We will now look at the definition of the 2nd-trampoline property.

Definition 4.0.6 *A distribution on the second trampoline graph that has some pebbles at v_N and $u_i = u^*$ and zero pebbles at every other vertex satisfies the 2nd-trampoline property at u^* if:*

- *The distribution is solvable for T.*

- *Any sequence of pebbling moves that reaches T has exactly 2 JF - points.*
- *With fewer pebbles at either v_N or u^*, the distribution is unsolvable.*

4.1 2nd-Trampoline Property Significant Pebbling Cases

Similar to the case of the 1st-trampoline property, there are four particular groups of $s^{-}_{@}(u^*) = x^*$ for $c \in \mathbb{N}, c < 2^j$ that are significant for the 2nd- trampoline property:

(1) $0 < x^* \leq 2^{d^*}(2^d - 2^{D-\hat{d}})$

(2) $2^{d^*}(c * 2^d - 2^{D-\hat{d}}) < x^* < c * 2^{d^*+d}$

(3) $c * 2^{d^*+d} \leq x^* \leq 2^{d^*}[(c+1) * 2^d - 2^{D-\hat{d}}]$

(4) $2^{d^*}(2^{d+j} - 2^{D-\hat{d}}) < x^* < 2^{d^*+d+j}$

Note that these ranges of x^* for the 2nd-trampoline property are very similar to the ranges of x for the 1st-trampoline property. The noticeable difference is that every range has an added 2^{d^*} to account for the extra cost of sending the pebbles along the path of length d^*. It is important to note that, if $x^* < 2^{d*}$, we could not reach u with this amount of pebbles at u^*, therefore the pebbles would not help us to reach T. When determining which values of

x^* satisfy the 2nd-trampoline property, it is also useful to only look at cases of x^* such that $\lfloor x^*/2^{d^*} \rfloor = x^*/2^{d^*}$, or in other words when x^* is divisible by 2^{d^*}. If this were not true, we would not satisfy the 2nd- trampoline property because, with fewer pebbles at u^*, the distribution would still be solvable.

The four ranges of x^* above inspire the following theorems.

Theorem 4.1.1 *If we let* $D(u^*) = x^*$*, where* $0 < x^* \leq 2^{d^*}(2^d - 2^{D-\hat{d}})$ *and* $D(v) = 0$, $\forall\ v \in V(G) - \{u^*, v_N\}$*, then* $D(v_N) = 2^N$ *is the smallest amount of pebbles required to reach* T.

Proof. This proof is identical to the proof of Theorem 3.1.1 above since, after sending x^* pebbles from u^* towards u, we end up with x pebbles at u, where $x \leq 2^d - 2^{D-\hat{d}}$. Hence we can follow the proof of Theorem 3.1.1. □

Note that the configuration for u^* described in Theorem 4.1.1 is too small to be beneficial in reaching T. In other words, when $x \leq 2^d - 2^{D-\hat{d}}$, more pebbles would need to be sent up the arch A than across the path R in any attempts to reach v_j. This means the smallest amount of pebbles needed at v_N to reach T is 2^N, the same amount of pebbles as is needed when $x^* = 0$. Additionally, this sequence has zero JF-points.

Therefore, if $D(v_N) = 2^N$ and $D(v) = 0$, $\forall\ v \in V(G) - \{v_N\}$, we can still reach T. Then $D(v_N) = 2^N$ is the smallest amount of pebbles needed to reach T

when $0 < x^* \leq 2^{d^*}(2^d - 2^{D-\hat{d}})$, but we can still reach T with fewer pebbles at u^*. Hence this configuration does not satisfy either the second or third requirements of the 2nd-trampoline property.

Theorem 4.1.2 *If we let* $D(u^*) = x^*$ *where* $2^{d^*}(c * 2^d) \leq x^* \leq 2^{d^*}[(c+1) * 2^d - 2^{D-\hat{d}}]$ *for* $c < 2^j, c \in \mathrm{N}$ *and* $D(v) = 0, \forall v \in V(G) - \{u^*, v_N\}$, *then* $D(v_N) = 2^{N-j}(2^j - c)$ *is the smallest amount of pebbles at* v_N *required to reach* T.

Proof. This proof is identical to the proof of Theorem 3.1.2 because sending all x^* pebbles at u^* towards u results in x pebbles arriving at u, where $c * 2^d \leq x \leq (c+1) * 2^d - 2^{D-\hat{d}}$. Therefore we can follow the proof of Theorem 3.1.2. □

One difference to note between Theorem 3.1.2 and Theorem 4.1.2 is that Theorem 4.1.2 allows for the case when $x^* = 2^{d^*}(c * 2^d)$, meaning $x = c * 2^d$. More will be said about this after Theorem 4.1.4.

Note also that the configuration described in Theorem 4.1.2 does not satisfy the 2nd-trampoline property. We can still reach T with $D(v_N) = 2^{N-j}(2^j - c/2^{d^*})$ and fewer pebbles at u^*, thus the third requirement of the 2nd-trampoline property is not met. Unlike the range of x^* described in Theorem 4.1.1, this range of x^* will have two JF-points (one at u and one at v_j), hence the second requirement of the 2nd-trampoline property is met.

Therefore, if $D(u^*) = x^*$, where $c * 2^{d^*+d} < x^* \leq 2^{d^*}[(c+1) * 2^d - 2^{D-\hat{d}}]$ for $c < 2^j, c \in \mathbb{N}$, then $2^{N-j}(2^j - \lfloor x^*/(2^{d+d^*})\rfloor)$ is the smallest amount of pebbles required at v_N to reach T, however this distribution does not satisfy the third requirement of the 2nd-trampoline property.

Theorem 4.1.3 *If we let $D(u^*) = x^*$ where $2^{d^*}(2^{d+j} - 2^{D-\hat{d}}) < x^* < 2^{d^*+d+j}$ and $D(v) = 0, \forall v \in V(G) - \{u^*, v_N\}$ then $D(v_N) = 2^{N-i}[2^{\hat{d}}(2^{d+j} - \lfloor x^*/2^{d^*}\rfloor)]$ is the smallest amount of pebbles required at v_N to reach T.*

Proof. This proof is identical to the proof of Theorem 3.1.3 because sending all x^* pebbles at u^* towards u results in x pebbles arriving at u, where $2^{d+j} - 2^{D-\hat{d}} < x \leq 2^{d+j}$. Therefore we can follow the proof of Theorem 3.1.3. □

Like the proof in Theorem 3.1.3, sending all pebbles from v_N up A is the most efficient use of the pebbles at u^*. But this sequence will have only one JF-point at u, therefore the second requirement of the 2nd-trampoline property is not met.

Therefore if $D(u^*) = x^*$, where $2^{d^*}(2^{d+j} - 2^{D-\hat{d}}) < x^* < 2^{d^*+d+j})$ then $D(v_N) = 2^{N-i}[2^{\hat{d}}(2^{d+j} - \lfloor x^*/2^{d^*}\rfloor)]$ is the smallest amount of pebbles required at v_N to reach T, however, this distribution does not satisfy the second requirement of the 2nd-trampoline property.

Theorem 4.1.4 *If we let* $D(u^*) = x^*$ *where* $2^{d^*}(c * 2^d - 2^{D-\hat{d}}) < x^* < c * 2^{d^*+d}$ *for* $c < 2^j, c \in \mathrm{N}$ *and* $D(v) = 0$, $\forall v \in V(G) - \{u^*, v_N\}$ *then specifying* $D(v_N) = 2^{N-i}[2^D(2^j - c) + 2^{\hat{d}}(c * 2^d - \lfloor x/2^{d^*} \rfloor)]$ *results in D satisfying the 2nd-trampoline property.*

Proof. Like the proof in Theorem 3.1.4, the only way to reach T given this distribution of pebbles is to send some of the pebbles from v_N up A and some across R, while utilizing the pebbles brought down from u^*.

The two JF-points are u and v_j.

With fewer pebbles at either v_N or u^*, the distribution would be unsolvable. This follows from part (3) of the proof for Theorem 3.1.4 since sending all x^* pebbles at u^* towards u results in x pebbles arriving at u, where $c * 2^d - 2^{D-\hat{d}} < x < c * 2^d$.

Therefore if we let $D(u^*) = x^*$ where $2^{d^*}(c*2^d - 2^{D-\hat{d}}) < x^* < c*2^{d^*+d}$ and $D(v) = 0, \forall v \in V(G) - \{u^*, v_N\}$, then $D(v_N) = 2^{N-i}[2^D(2^j - c) + 2^{\hat{d}}(c*2^d - \lfloor x/2^{d^*} \rfloor)]$ pebbles at v_N satisfies the 2nd-trampoline property. ◻

It is important to point out that, unlike the range for x that satisfies the 1st-trampoline property, the range for x^* that satisfies the 2nd-trampoline property is strictly less than $c * 2^{d^*+d}$. This is because, when $x^* = c * 2^{d^*+d}$, it is most efficient to send all of the pebbles from u^* down A to join forces at v_j

with the pebbles from v_N that are all sent across R, resulting in only one JF-point. Therefore when $x^* = c * 2^{d^*+d}$, our distribution does not satisfy the 2nd-trampoline property.

4.2 2nd-Trampoline Property Examples

Recall that there are four significant cases for u^*. These are:

(1) $0 < x^* \leq 2^{d^*}(2^d - 2^{D-\hat{d}})$

(2) $2^{d^*}(c * 2^d - 2^{D-\hat{d}}) < x^* < c * 2^{d^*+d}$

(3) $c * 2^{d^*+d} \leq x^* \leq 2^{d^*}[(c+1) * 2^d - 2^{D-\hat{d}}]$

(4) $2^{d^*}(2^{d+j} - 2^{D-\hat{d}}) < x^* < 2^{d^*+d+j}$

Consider $d = 3,\ \hat{d} = 4,\ d^* = 2,\ D = 5,\ j = 2,\ N - i = 2,\ and\ c = 1.$

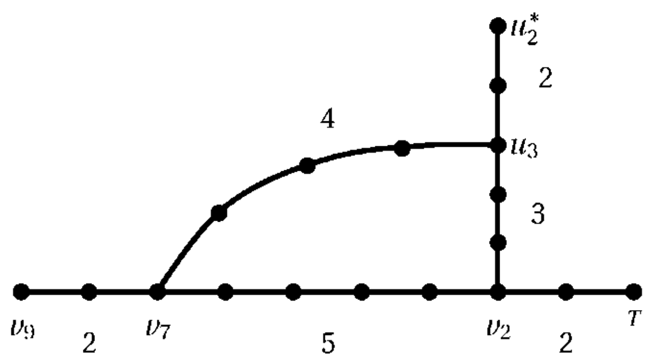


Figure 4.2: Example of the 2nd-Trampoline Graph.

Similar to Section 3.2, the 4 significant cases for x^* are:

(1) $0 < x^* \leq 24$

(2) $24 < x^* < 32, 56 < x < 64, 88 < x < 96$

(3) $32 \leq x^* \leq 56, 64 \leq x \leq 88, 96 \leq x \leq 120$

(4) $120 < x^* < 128$

Because it is only useful to look at cases of x^* that are multiples of 2^{d^*}, we will only consider values of x^* that are multiples of 4. In other words, the four significant cases for x^* become:

(1) $x^* = 4, 8, 12, 16, 20, 24$

(2) $x^* = 28, 60, 92$

(3) $x^* = 32, 36, 40, 44, 48, 52, 56, 64, 68, 72, 76, 80, 84, 88, 96, 100, 104, 108, 112, 116, 120$

(4) $x^* = 124$

Example 4.2.1 *If we let $D(u^*) = x^*$, where $x^* = 4, 8, 12, 16, 20, 24$, it will always be most cost efficient to send all pebbles from v_9 straight across R. Hence, we will always place 512 pebbles at v_9 in order to reach T.*

If we want to send some pebbles up A and some across R, we need at least 704 pebbles at v_9 to reach T.

$$v_9 \xrightarrow[R]{704} v_7 \quad \text{gives} \quad P_{\rightarrow}(v_7) = 176$$

$$v_7 \xrightarrow[R]{96} v_2 \quad \text{gives} \quad P_{\rightarrow}(v_2) = 3$$

$$v_7 \xrightarrow[A]{80} u_3 \quad \text{gives} \quad P_{\rightarrow}(u_3) = 5$$

$$u_2^* \xrightarrow[A]{12} u_3 \quad \text{gives} \quad P_{\rightarrow}(u_3) = 3$$

$$u_3 \xrightarrow[R]{8} v_2 \quad \text{gives} \quad P_{\rightarrow}(v_2) = 1$$

$$v_2 \xrightarrow[R]{4} T \quad \text{gives} \quad P_{\rightarrow}(T) = 1$$

Just like Example 3.2.1, with only $2^9 = 512$ pebbles at v_9, we can reach T directly by sending all 512 pebbles across R. This means that this distribution of x^* does not satisfy the 2nd-trampoline property because there are zero JF-points with this configuration.

Example 4.2.2 *If we let $D(u^*) = x^*$, where $x^* = 28, 60, 92$, then we will always satisfy the 2nd-trampoline property of the graph.*

Let $D(v_9) = 320$ and $D(u^*) = 60$. Then if we want to send some pebbles up A and some across R,

$$v_9 \xrightarrow[R]{320} v_7 \quad \text{gives} \quad P_{\rightarrow}(v_7) = 80$$

$$v_7 \xrightarrow[R]{64} v_2 \quad \text{gives} \quad P_{\rightarrow}(v_2) = 2$$

$$v_7 \xrightarrow[A]{16} u_3 \quad \text{gives} \quad P_{\rightarrow}(u_3) = 1$$

$$u_2^* \xrightarrow[A]{60} u_3 \quad \text{gives} \quad P_{\rightarrow}(u_3) = 15$$

$$u_3 \xrightarrow[R]{16} v_2 \quad \text{gives} \quad P_{\rightarrow}(v_2) = 2$$

$$v_2 \xrightarrow[R]{4} T \quad \text{gives} \quad P_{\rightarrow}(T) = 1$$

Just like Example 3.2.2, if we instead send all of our pebbles up A, we would need a total of 1088 pebbles at v_9. Sending all pebbles across R would again require 512 pebbles at v_9, therefore it is most cost efficient to send some pebbles up A and some across R.

Example 4.2.3 *If we let* $D(u^*) = x^*$, *where* $x^* = 32, 36, 40, 44, 48, 52, 56, 64, 68, 72, 76, 80, 84, 88, 96, 100, 104, 108, 112, 116, 120$, *then it will always be most cost efficient to send all of our pebbles from* v_9 *across* R *in order to reach* T.

Let $D(v_9) = 128$ and $x^* = 100$. If we want to send some pebbles up A and some across R, we would need to send at least 112 pebbles up A from v_7 in order for 4 pebbles to reach v_2. In this particular case, no pebbles would need to be sent across R, therefore a total of 448 pebbles would be required at v_9. But if we send all pebbles across R,

$$v_9 \xrightarrow[R]{128} v_2 \quad \text{gives} \quad P_{\rightarrow}(v_2) = 1$$

$$u_2^* \xrightarrow[A]{100} u_3 \quad \text{gives} \quad P_{\rightarrow}(u_3) = 25$$

$$u_3 \xrightarrow[R]{25} v_2 \quad \text{gives} \quad P_{\rightarrow}(v_2) = 3$$

$$v_2 \xrightarrow[R]{4} T \quad \text{gives} \quad P_{\rightarrow}(T) = 1$$

This range of pebbles at x^* has 2 JF-points and reaches T, but with only 96 pebbles at u_2^* we could still reach T, therefore this distribution does not satisfy the 2nd-trampoline property.

Example 4.2.4 *If we let* $D(u^*) = x^*$, *where* $x^* = 124$, *then it will always be most cost efficient to send all of our pebbles from* v_9 *up the arch* A *to reach* T.

If we want to send all pebbles from v_9 across the path, we would need at least 128 pebbles at v_9. It does not make sense to send some up the arch and some across the path because we only need one more pebble to reach v_2 in order to reach T, therefore there is no reason to split the pebbles.

If we send all pebbles up the arch, only 64 pebbles are required to be placed at v_9, therefore this is the most cost efficient method. This range of pebbles at x has zero JF-points, therefore we do not satisfy the 2nd-trampoline property.

CHAPTER 5

FIRST TRAMPOLINE PEBBLING NUMBER

As mentioned at the beginning of this thesis, $\pi(G)$ is unknown for most families of graphs. The trampoline graphs introduced in this paper are included in that list of graphs whose pebbling number is unknown. What makes finding the pebbling number of a trampoline graph even more difficult is the fact that trampoline graphs are NOT greedy. This is because the pebbling strategy of the graph often requires utilizing an arch of the graph whose vertices are no closer to the target vertex T. At best, when $D - d - \widehat{d} = 1$, a trampoline graph is *semi-greedy* [2], meaning every configuration of size at least $\pi(G)$ has a *semi-greedy solution*, in which every pebbling step is a step from a vertex v_i to a vertex v_j such that $d(v_j, v_0) \leq d(v_i, v_0)$, where v_0 is the target vertex. This means no pebbling moves take us farther from the target vertex.

In this chapter we will provide lower bounds for general cases of the pebbling number of the 1st-trampoline graph. We will also prove the pebbling number for two specific cases.

5.1 Pebbling Number Estimates for the 1st Trampoline Graph

Theorem 5.1.1 *Let G be a first trampoline graph. When $d + \hat{d}$ is EVEN:* $\pi(G) \geq 2^N + 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}}$.

Proof. We will first prove that there exists a T-unsolvable distribution on G of size $2^N + 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}} - 1$.

Place $2^N - 1$ pebbles at v_N and $2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}}$ pebbles at a vertex u on the arch A at a distance $\frac{d+\hat{d}}{2}$ from v_j, as in the figure below. We will now prove (by contradiction) that the vertex T cannot be reached. Suppose otherwise. Necessarily our first step is to send the pebbles from v_N towards T by sending

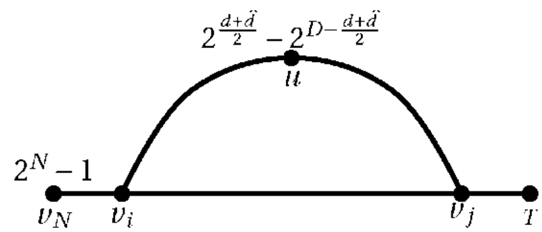


Figure 5.1: Unsolvable distribution for $d + \hat{d}$ even.

as many as possible to v_i. Observe that

$$v_N \xrightarrow[R]{2^N - 1} v_i \quad \text{gives} \quad p_{\rightarrow}(v_i) = 2^i - 1.$$

Now we observe that $2^i - 1$ pebbles at v_i is not enough to reach T if we send all of our pebbles directly across the path R. We also observe that $2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}}$ pebbles at u is not enough to reach v_j (since $d(u, v_j) = d$ and $D > d$, therefore $2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}} < 2^d$).

If we attempt to send all $2^i - 1$ pebbles up the arch A, we can send at most $2^{i-\hat{d}} + 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}}$ pebbles from u towards v_j, so that $2^{i-\hat{d}-d} + 2^{\frac{d+\hat{d}}{2}-d} - 2^{D-\frac{d+\hat{d}}{2}-d} = 2^{i-\hat{d}-d} + 2^{\frac{\hat{d}-d}{2}} - 2^{D-\frac{\hat{d}-d}{2}}$ pebbles reach v_j, where $2^{i-\hat{d}-d} < 2^j$ and $2^{\frac{\hat{d}-d}{2}} < 2^{D-\frac{\hat{d}-d}{2}}$. This means $2^{i-\hat{d}-d} + 2^{\frac{\hat{d}-d}{2}} - 2^{D-\frac{\hat{d}-d}{2}} < 2^j$, and so we still cannot reach T.

Therefore the only remaining way to reach T with this configuration of pebbles is to send some of the pebbles from v_i up A and some across R. This makes v_j a joining forces point.

If we send all $2^i - 1$ pebbles from v_i to v_j, at most $2^j - 1$ pebbles will arrive. Thus if we send only $2^D(2^j - 1)$ across the path, we can still reach v_j while at the same time not wasting any pebbles unnecessarily. This leaves us with $2^D - 1$ pebbles that can be sent up the arch. We observe that

$$v_i \xrightarrow[A]{2^D - 1} u \quad \text{gives} \quad p_{\to}(u) = \lfloor \frac{2^D - 1}{2^{\frac{d+\hat{d}}{2}}} \rfloor = 2^{D-\frac{d+\hat{d}}{2}} - 1.$$

Since $s^-_{@}(u) = 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}}$, we have $s^+_{@}(u) = 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}} + 2^{D-\frac{d+\hat{d}}{2}} - 1 = 2^d - 1$, which is not enough to reach v_j. This is a contradiction to the assumption that we could reach T, which means we have found a T-unsolvable distribution of size $2^N - 1 + 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}}$ on the 1st-trampoline graph. However, if we add a single pebble to any vertex, we are able to reach T.

Hence, $\pi(G) \geq 2^N + 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}}$. □

Unfortunately, we cannot prove that $\pi(G) = 2^N + 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}}$ in every instance because there are particular graphs for which this is not the case.

Example 5.1.2 *Let $N = 7$, $j = 1$, $N - i = 1$, $d + \hat{d} = 6$. Then the following figures show $\pi(G) \neq 2^N + 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}}$.*

In the figure below, we have an unsolvable distribution of $2^N + 2^{\frac{d+\hat{d}}{2}} - 2^{D-\frac{d+\hat{d}}{2}} - 1 = 2^7 + 2^3 - 2^2 - 1 = 131$ pebbles.

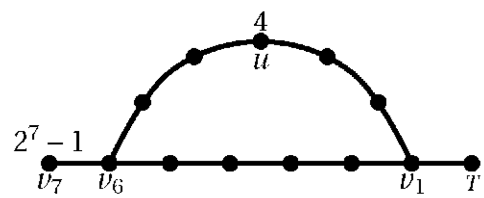


Figure 5.2: Example of an unsolvable distribution.

But it is possible to add another pebble on the arch such that the distribution is still unsolvable, as is shown in the following figure.

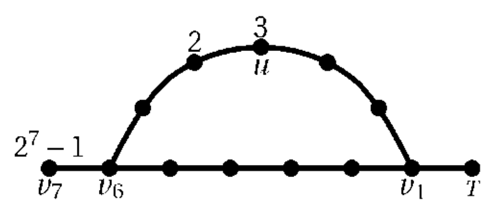


Figure 5.3: Example of a larger unsolvable distribution.

Therefore, it is possible to create an unsolvable distribution of this particular trampoline graph using $2^N + 2^{\frac{d+\widehat{d}}{2}} - 2^{D-\frac{d+\widehat{d}}{2}}$ pebbles, meaning that for this graph, $\pi(G) \geq 2^N + 2^{\frac{d+\widehat{d}}{2}} - 2^{D-\frac{d+\widehat{d}}{2}} + 1$.

Theorem 5.1.3 *Let G be a first trampoline graph. When $d + \widehat{d}$ is ODD, $\pi(G) \geq 2^N + 2^{\frac{d+\widehat{d}-1}{2}} - 1$.*

Proof. We will first prove that there exists a configuration on our graph G consisting of $2^N + 2^{\frac{d+\widehat{d}-1}{2}} - 1$ pebbles such that we cannot reach T. Select a vertex u on the arch A at a distance $\frac{\hat{d}+d+1}{2}$ from v_j. Place $2^N - 1$ pebbles at v_N and $2^{\frac{d+\hat{d}-1}{2}} - 1$ pebbles on u. Similar to the argument in the previous proof, we need to send at most $2^D - 1$ pebbles from v_i along A so that these arriving pebbles combine with the pebbles at u to reach v_j and create a joining forces point with the pebbles that are send towards v_j along R.

Note that, as before, $d \leq \hat{d} \leq d+1$, where $d(u, v_j) = d$ and $d(v_i, u) = \hat{d}$. In fact, since $d + \hat{d}$ is odd, $\hat{d} = d + 1$.

When $d + \hat{d}$ is odd, we can place more pebbles at the $u_{\frac{d+\hat{d}+1}{2}}$ vertex (which we can simplify as u_{d+1}) without reaching R than at any other vertex on A.

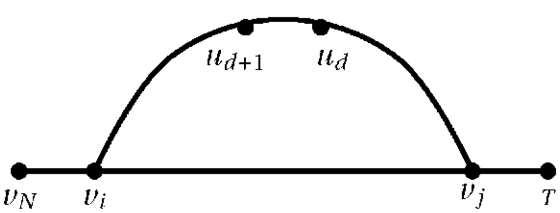


Figure 5.4: u_{d+1} and u_d at farthest distance from R.

The maximum number of pebbles that can be placed at this vertex without reaching v_i is $2^{\frac{d+\hat{d}-1}{2}} - 1$, i.e. $2^d - 1$. Similar to the case when $d + \hat{d}$ is even, we observe that

$$v_i \xrightarrow[A]{2^D - 1} u \quad \text{gives} \quad p_{\rightarrow}(u_{d+1}) = 2^{D-d} - 1.$$

This means that the total pebbles sitting at the u_{d+1} vertex after the sequence of pebbling moves from v_N is $s^+_@(u_{d+1}) = 2^d - 1 + 2^{D-d} - 1 = 2^d + 2^{D-d} - 2$, which in not enough to reach v_j.

Therefore there exists a configuration of pebbles on our graph such that with $2^N + 2^d + 1$ pebbles, we cannot reach T. Hence, $\pi(G) \geq 2^N + 2^{\frac{d+\hat{d}-1}{2}} - 1$. □

We can take this proof one step further than we could when $d+\widehat{d}$ was even. If we recall back to the point of our proof at which $s^+_@(u_{d+1}) = 2^d + 2^{D-d} - 2$, we can track the pebbles along their journey to v_j and make some more discoveries.

The number of pebbles that arrive to the adjacent vertex, u_d, is

$p_\rightarrow(u_d) = \lfloor \frac{2^d + 2^{D-d} - 2}{2} \rfloor = 2^{d-1} + 2^{D-d-1} - 1$, so that

$p_\rightarrow(u_{d-1}) = \lfloor \frac{2^{d-1} + 2^{D-d-1} - 1}{2} \rfloor = 2^{d-2} + 2^{D-d-2} - 1$ pebbles will arrive to the next vertex. We notice that if $p_\rightarrow(u_{d-1}) < 2^{d-1} - 1$, we can allow pebbles to sit at the u_{d-1} vertex such that $s^-_@(u_{d-1}) + p_\rightarrow(u_{d-1}) = 2^{d-1} - 1$. Note that $s^-_@(u_{d-1}) \leq 3$, otherwise those pebbles could combine with the $s^-_@(u_{d+1})$ pebbles to reach v_i.

This pattern can be continued as we approach v_j such that at most 3 pebbles can be added at every other vertex along the arch. Therefore, for large enough values of $d + \widehat{d}$ odd, $\pi(G) \geq 2^N + 2^d + \lfloor \frac{3(d-2)}{2} \rfloor$.

As the difference between $d+\hat{d}$ and D increases, $\pi(G)$ also increases. While $2^N + 2^d - 1$ is the lower bound for the size of the maximal unsolvable distribution for all $d, \hat{d}$, and D, this bound will increase as $d + \hat{d} - D$ grows.

5.2 Pebbling Number Proofs for Particular 1st Trampoline Graphs

Theorem 5.2.1 34 *is the pebbling number of the following graph.*

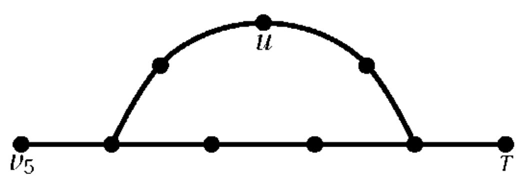


Figure 5.5: 1st- trampoline graph with pebbling number of 34.

Proof. From Remark 2.1.3 in Chapter 2, we know that $2^N + 2^{\frac{(d+\hat{d})}{2}} - 2^{\frac{D-(d+\hat{d})}{2}} \geq 2^N + 2$. Combining this with Theorem 5.1.1, we know the pebbling number of the graph shown in Figure 5.5 is at least 34.

The following distribution of 33 pebbles is unsolvable, but by adding a single pebble anywhere to the graph, it becomes possible to reach T.

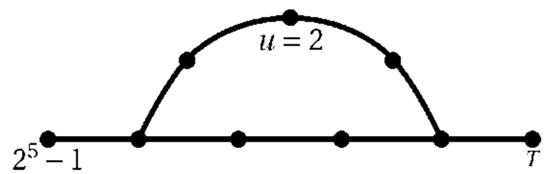


Figure 5.6: Unsolvable distribution of 33 pebbles.

To prove that 34 is indeed the pebbling number of this trampoline graph, let us first assume our target vertex T is on the path under the arch or on the arch itself. This means that splitting the graph in half as in the image below would result in at least one of the splits containing at least 17 pebbles. Without loss of generality, assume T is on the left side of the split.

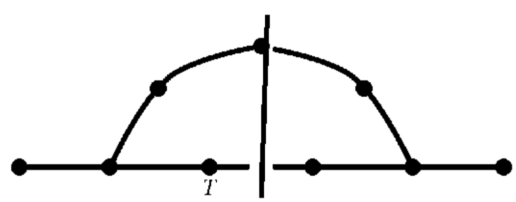


Figure 5.7: Split graph with T on the left.

The longest path to T from any other vertex is of length 3, which only requires 8 pebbles, therefore it is more than possible to reach T with a distribution of 34 pebbles on the graph when it is either on the arch or under it.

This means that the only remaining potential vertices for T are on one of the outside legs of the path R, i.e. either v_5 or v_0. Then we can assume without loss of generality that $T = v_0$.

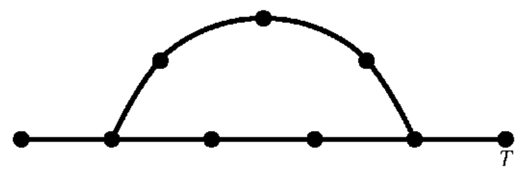


Figure 5.8: Graph with $T = v_0$.

If we can reach v_0 with any distribution of size 34, we will have proven that 34 is the pebbling number of this 1st trampoline graph.

Now, if there are 34 total pebbles on the graph, this means that either the arch or the path has at least 16 pebbles. Assume the arch has 16 pebbles, then we can reach T and we are done.

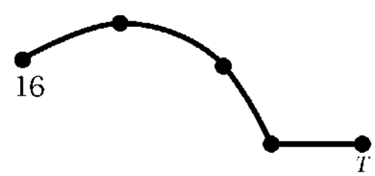


Figure 5.9: $\pi(P_5) = 16$.

Then assume the arch has at most 15 pebbles. Therefore assume the path has at least 19 pebbles.

If the path has 32 pebbles, we're done.

Figure 5.10: $\pi(P_6) = 32$.

Thus assume the path has at most 31 pebbles. Hence the path has $19 \leq x \leq 31$ pebbles and the arch has $3 \leq y \leq 15$.

With only 16 pebbles on the path, we can reach v_1 once, so we only need to reach v_1 once more with the remaining $\widehat{x}$ pebbles in order to reach T. Then the path has $3 \leq \widehat{x} \leq 15$ pebbles and the arch has $3 \leq y \leq 15$.

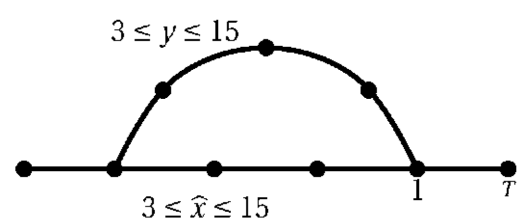


Figure 5.11: $3 \leq \widehat{x} \leq 15$ and $3 \leq y \leq 15$.

(a) Let $y \geq 8$. Then we have enough pebbles on the arch to be able to reach v_1 again, leaving at most 7 pebbles on the arch and 11 pebbles on the path.

When $\hat{x} \geq 11$, no pebbles can be placed on the v_2 vertex. If at least 2 pebbles are placed on the vertex, we can reach v_1 at least once. If 1 pebble is placed on the v_2 vertex, we can then reach v_2 again with the minimum 10 pebbles placed in any configuration between v_3 and v_5. Then zero pebbles are on the v_2 vertex and at least 11 pebbles are on the v_3, v_4, and v_5 vertices.

When $\hat{x} \geq 11$, at most 1 pebble can be placed on the v_3 vertex. If at least 4 pebbles are placed on the vertex, we can reach v_1 at least once. If 3 pebbles are placed on the v_3 vertex, we can reach v_3 again with the minimum 8 pebbles between v_4 and v_5. If 2 pebbles are placed on the v_3 vertex, we can reach v_3 twice with the minimum 9 pebbles between v_4 and v_5. If 1 pebble is placed on the v_3 vertex, it is possible to reach v_3 at most two times with the minimum 10 pebbles placed in any configuration between v_4 and v_5. Then at most 1 pebble is placed on the v_3 vertex and at least 10 pebbles are on the v_4, and v_5 vertices.

When $\hat{x} \geq 13$, zero pebbles can be placed on the v_3 vertex. If at least 4 pebbles are placed on the vertex, we can reach v_1 at least once. If 3 pebbles are placed on the v_3 vertex, we can reach v_3 again with the minimum 10 pebbles between v_4 and v_5. If 2 pebbles are placed on the v_3 vertex, we can reach v_3 twice with the minimum 11 pebbles between v_4 and v_5. If 1 pebble is placed on the v_3 vertex, we can reach v_3 three times with the minimum 12 pebbles placed in any configuration between v_4 and v_5. Then zero pebbles are on the v_3 vertex and at least 13 pebbles are on the v_4, and v_5 vertices.

(b) $y = 3$ and $\hat{x} = 15$. Then all 15 pebbles on the path must be placed at v_5, otherwise we can reach v_1.

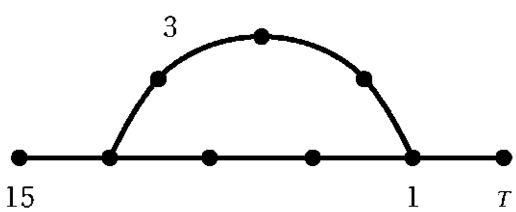
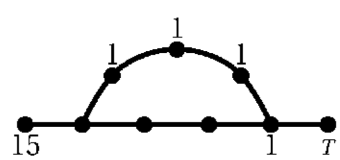


Figure 5.12: $\hat{x} = 15$ and $y = 3$.

The only configurations of $y = 3$ that do not reach v_1 or v_4 are shown in the following graphs.



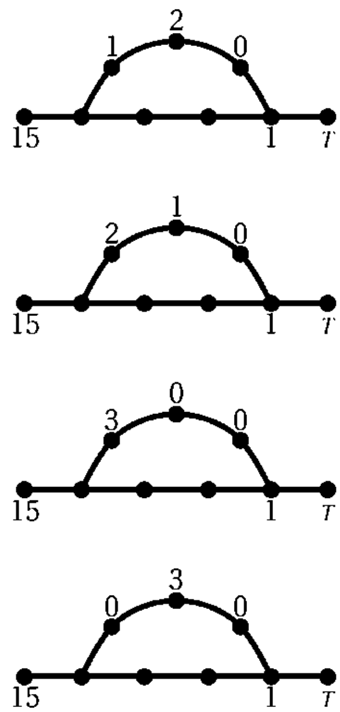


Figure 5.13: Five configurations of 3 pebbles on the arch that do not reach R.

Sending all of the 15 pebbles from the path up the arch, at least 3 arrive at u_3. With the five configurations of pebbles on the arch above, plus the 3 additional pebbles from the path, it is always possible to reach v_1 by sending all pebbles across A.

(c) $y = 4$ and $\widehat{x} = 14$. Then in order to not reach v_1, all 4 pebbles on the arch must be placed in such a way that they have to reach v_4. This can be shown by attempting to add a single pebble to any of the five graphs above. So we can reach v_4 with at least 1 pebble.

Of the 14 pebbles already on the path between v_4 and v_5, at least 7 pebbles will arrive at v_4, plus the 1 pebble from the arch. Then with 8 pebbles at v_4, we can reach v_1 by sending all 8 of those pebbles across the path.

(d) $y = 5$ and $\hat{x} = 13$. With at least 5 pebbles on the arch, u_1 cannot have any pebbles, otherwise with at least 4 pebbles between u_2 and u_3, we could reach v_1. With 4 pebbles on u_2, we can reach v_1. With 3 pebbles on u_2, we would have 2 pebbles on u_3, therefore we could reach v_1 By sending both pebbles from u_3 to u_2 and sending the resulting 4 pebbles from u_2 to v_1. Therefore u_2 has at most 2 pebbles and u_3 has at least 3 and at most 5 pebbles.

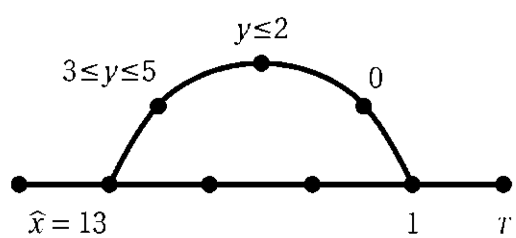


Figure 5.14: At most 2 pebbles on u_2 and between 3 and 5 pebbles on u_3.

With 13 pebbles between v_4 and v_5, at least 6 pebbles arrive at v_4, so at least 3 pebbles can arrive at u_3. This means we now have at least 8 pebbles between u_2 and u_3. With this amount of pebbles, we can reach v_1 by sending all of them across the arch.

We can also send the pebbles from the arch down. With at most 2 pebbles on u_2 and $3 \leq y \leq 5$ pebbles on u_3, at least 2 pebbles will arrive to v_4.

Of the 13 pebbles already on the path between v_4 and v_5, at least 6 pebbles will arrive at v_4, plus the 2 pebbles from the arch. Then with 9 pebbles at v_4, we can reach v_1 by sending all 8 of those pebbles across the path.

(e) $y = 6$ and $\hat{x} = 12$. From above, at most 1 pebble sits at v_3 when $\hat{x} = 12$, so worst case is that 1 pebble is at v_3 and 11 pebbles sit at v_5. This means 5 pebbles arrive at v_4 from v_5, so at least 2 pebbles arrive at u_3 from v_4. With 6 pebbles on the arch that do not reach v_1 on their own, zero pebbles can sit at u_1, at most 1 pebble sits at u_2.

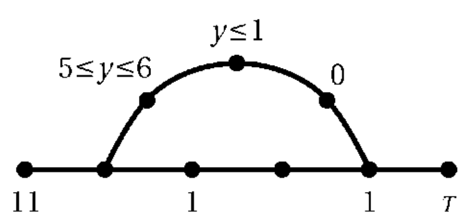


Figure 5.15: $y = 6$ and $\hat{x} = 12$.

Then when the 2 pebbles arrive at u_3 from v_4, there are a total of 8 pebbles between u_2 and u_3. With this we can reach v_1.

If we want to send the pebbles on the arch down, at least 2 pebbles will arrive to v_4 from u_3. Worst case on the path is to place all 12 pebbles on the v_5 vertex, therefore at least 6 pebbles reach v_4 plus the 2 from the arch for a total of 8 pebbles. With this, we can reach v_1.

(f) $y = 7$ and $\hat{x} = 11$. With 7 pebbles on the arch, the only configuration that will not reach v_1 is placing all pebbles on u_3. At least 5 pebbles can reach v_4 from the path, so at least 2 pebbles will reach u_3 from the path.

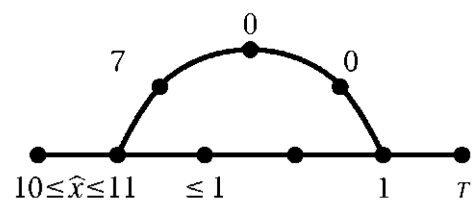


Figure 5.16: $y = 7$ and $\hat{x} = 11$.

Combining these 2 with the 7 pebbles on u_3, we can reach v_1.

If we instead bring the 7 pebbles from the arch down, 3 pebbles would arrive at v_4. At least 5 pebbles would reach v_4 from v_5, then we can again reach v_1.

Therefore any possible configuration of 34 pebbles on this graph reaches T. This means that 34 is the smallest possible number such that we can always reach T for any vertex T, i.e. 34 is the pebbling number of this graph. □

Theorem 5.2.2 67 *is the pebbling number of the following graph.*

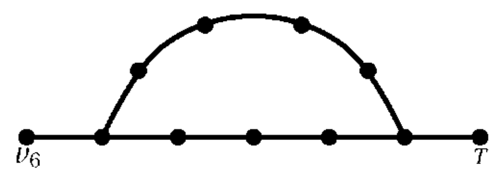


Figure 5.17: 1st-trampoline graph with pebbling number 67.

Proof. Note that with only 66 pebbles, the distribution in the graph below is unsolvable. But by adding a single pebble anywhere to the graph, it becomes possible to reach T.

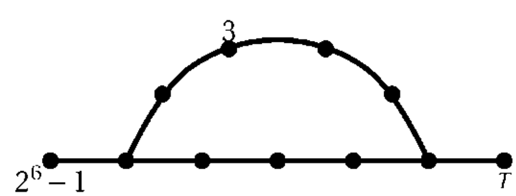


Figure 5.18: Unsolvable distribution of 66 pebbles.

To prove this, let us first assume our target vertex T is on the path under the arch or on the arch itself. This means that splitting the graph in half as in the image below would result in at least one of the splits containing at least 32 pebbles. Without loss of generality, assume T is on the left side of the split.

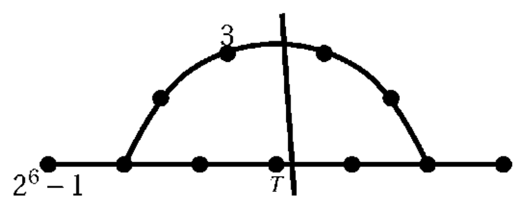


Figure 5.19: Split graph with T on the left.

The longest path to T from any other vertex is of length 4, which only requires 16 pebbles, therefore it is more than possible to reach T with a distribution of 67 pebbles when it is either on the arch or under it.

This means that the only remaining potential vertices for T are on one of the outside legs of the path R, i.e. either v_6 or v_0. Then we can assume without loss of generality that $T = v_0$.

Now, if there are 67 total pebbles on the graph, this means that either the arch or the path has at least 32 pebbles.

Assume the arch has 32 pebbles, then we can reach T and we are done. Then assume the arch has $y \leq 31$ pebbles. and the path has $x \geq 36$ pebbles.

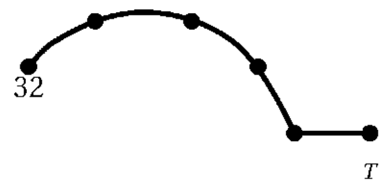


Figure 5.20: $\pi(P_6) = 32$.

If the path has 64 pebbles, we're done. Then assume $x \leq 63$.

Figure 5.21: $\pi(P_7) = 64$.

Therefore the path has $36 \leq x \leq 63$ pebbles and the arch has $4 \leq y \leq 31$. With only 32 pebbles on the path, we can reach v_1 once, so we only need to reach v_1 once more in order to reach T. Therefore the path has $4 \leq \hat{x} \leq 31$ pebbles and the arch has $4 \leq y \leq 31$.

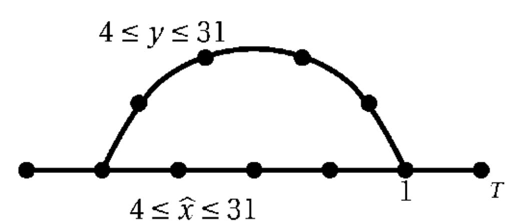


Figure 5.22: $4 \leq \hat{x} \leq 31$ and $4 \leq y \leq 31$.

(a) $y \geq 16$, then we have enough pebbles on the arch to be able to reach v_1 again. Then the arch has $4 \leq y \leq 15$ pebbles and the path has $20 \leq \hat{x} \leq 31$ pebbles.

When $\hat{x} \geq 20$, no pebbles can be placed on the v_2 vertex. If at least 2 pebbles are placed on the vertex, we can reach v_1 at least once. If 1 pebble is placed on the v_2 vertex, we can then reach v_2 again with at least 19 pebbles placed in any configuration between v_3 and v_6. Then zero pebbles are on the v_2 vertex and at least 20 pebbles are on the v_3, v_4, and v_5 vertices.

When $\hat{x} \geq 20$, at most 1 pebble can be placed on the v_3 vertex. If at least 4 pebbles are placed on the vertex, we can reach v_1 at least once. If 3 pebbles are placed on the v_3 vertex, we can reach v_3 again with at least 7 pebbles between v_4 and v_6. If 2 pebbles are placed on the v_3 vertex, we can reach v_3 twice with at least 18 pebbles between v_4 and v_6. If 1 pebble is placed on the v_3 vertex, it is possible to reach v_3 only twice with at least 19 pebbles placed in any configuration between v_4 and v_9. At most 1 pebble is placed on the v_3 vertex and at least 19 pebbles are on the v_4, v_5, and v_6 vertices.

When $\hat{x} \geq 25$, zero pebbles can be placed on the v_3 vertex. If at least 4 pebbles are placed on the vertex, we can reach v_1 at least once. If 3 pebbles are placed on the v_3 vertex, we can reach v_3 again with the minimum 22 pebbles between v_4 and v_6. If 2 pebbles are placed on the v_3 vertex, we can reach v_3 twice with the minimum 23 pebbles between v_4 and v_6. If 1 pebble is placed on the v_3 vertex, we can reach v_3 three times with the minimum 24 pebbles placed in any configuration between v_4, v_5, and v_6. Then zero pebbles are on the v_3 vertex and at least 25 pebbles are on the v_4, through v_6 vertices.

When $\hat{x} \geq 26$, at most 1 pebble can be placed at the v_4 vertex. If at least 8 pebbles are placed on the vertex,we can reach v_1 at least once. If 7 pebbles are placed on the vertex, we can reach v_4 again with the minimum 19 pebbles between v_5 and v_6. If 6 pebbles, we can reach twice with the minimum 20 pebbles, if 5, we can reach 3 times with the minimum 21 pebbles. If 4 pebbles are on v_4, we can reach it 4 more times with the 22 pebbles between v_5 and v_6, if 3, we can reach 5 more times with 23, if 2 we can reach 6 more times with 24 pebbles. If 1 pebble is on the v_4 vertex, we can only reach it 6 more times with the 25 pebbles between v_5 and v_6. Then at most one pebble is on the v_4 vertex when $\hat{x} \geq 26$.

When $\hat{x} \geq 29$, zero pebbles can be placed at the v_4 vertex. If 1 pebble is on the v_4 vertex, we can only reach it 7 more times with the 28 pebbles between v_5 and v_6. Then zero pebbles are on the v_4 vertex when $\hat{x} \geq 29$.

(b) $y = 4$ and $\hat{x} = 31$. Then all 31 pebbles on the path must be placed at v_6, otherwise we can reach v_1.

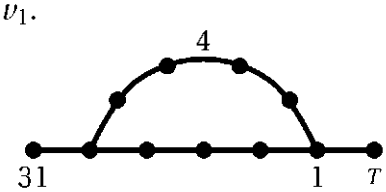
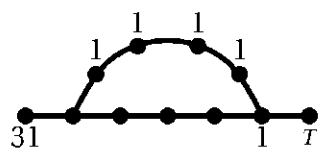
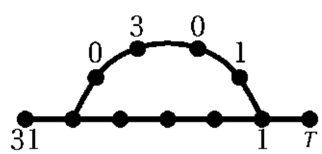
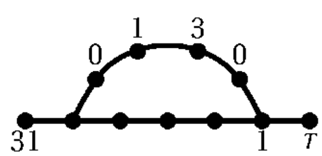


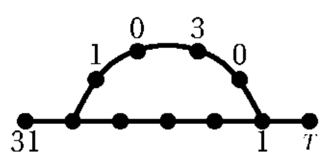
Figure 5.23: $\hat{x} = 31$ and $y = 4$.

The only configurations of $y = 4$ on the arch that do not reach v_1 or v_5 are the following.









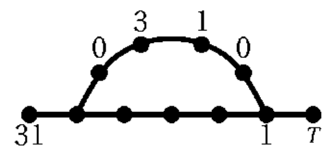
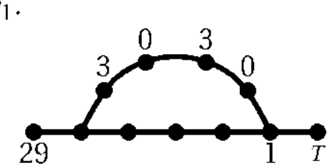


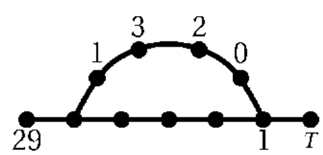
Figure 5.24: Five configurations of 4 pebbles on the arch that do not reach R.

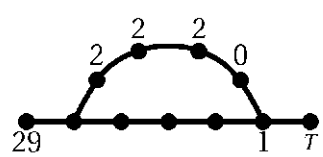
Sending all of the 31 pebbles from the path up the arch, at least 7 arrive at u_4. With these 7 additional pebbles from the path, it is always possible to reach v_1 by sending all pebbles across A. At most 1 pebble can sit on the u_1 vertex, otherwise we could reach v_1. When u_1 has a pebble, there are now $3+7$ pebbles between vertices u_2 through u_4. This allows us to reach u_1 once more. When u_1 has no pebbles, u_2 has at most 2 pebbles. Then with $2+7$ pebbles between u_3 and u_4, we can reach u_2 twice more.

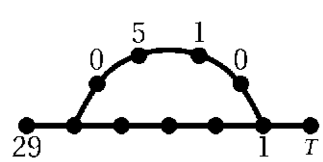
(c) $y = 5$ and $\hat{x} = 30$. Then in order to not reach v_1, all 5 pebbles on the arch must be placed in such a way that they have to reach v_5 with at least 1 pebble. This can be shown by attempting to add a single pebble to any of the five graphs above. The worst scenario again is if all 30 pebbles are placed at v_6, which means 15 pebbles arrive at v_5 from v_6 plus 1 pebble from u_4 will allow us to reach v_1 with a single pebble.

(d) $y = 6$ and $\hat{x} = 29$. As shown above, the 29 pebbles on the path will only sit on v_5 and v_6. This means that, at worst, 14 pebbles will arrive to v_5 from v_6. With 6 pebbles on the arch, either 1 or 2 pebbles will arrive to the path. In the event that 2 pebbles arrive from the arch, v_5 would then have $14+2$ pebbles and we would be done. The following graphs are the only configurations of graphs in which, with 6 pebbles on the arch, at most 1 pebble may arrive at v_5 and zero would arrive at v_1.









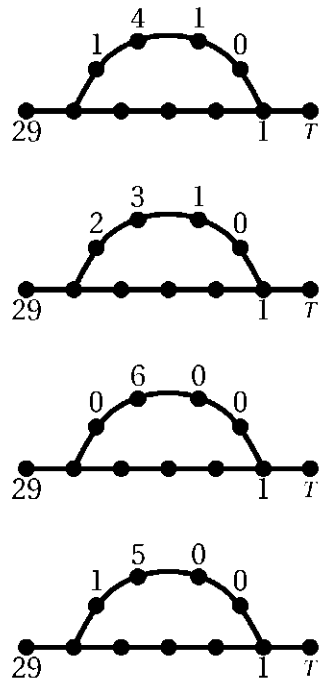


Figure 5.25: Eight configurations of 6 pebbles on the arch.

In these cases, send the 14 pebbles up the arch from v_5 instead so that 7 pebbles arrive to u_4. In all of these cases, we can then reach v_1.

(e) $y = 7$ and $\hat{x} = 28$. With 7 pebbles on the arch, either 1, 2, or 3 pebbles arrive to v_5. If either 2 or 3 arrive, we can reach v_1 by sending all of the 28+ either 2 or 3 pebbles across the path because the worst case puts all 28 pebbles on v_6, so 14 pebbles would arrive to v_5, plus either 2 or 3 gives us 16 or 17. From above, when $\hat{x} = 28$, at most 1 pebble can be placed on v_4, so the worst case would send 13 pebbles up the arch for 6 to arrive at u_4. The following graphs are the only configurations of graphs in which, with 7 pebbles on the arch, at most 1 pebble may arrive at v_5.

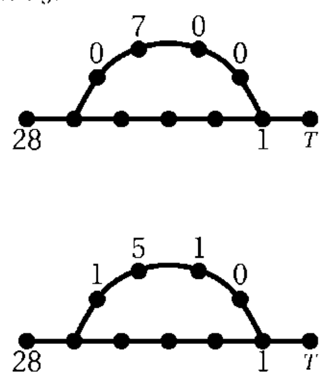


Figure 5.26: Two configurations of $y = 7$ in which 1 pebble reaches v_5.

In both cases, adding 6 pebbles to u_4 allows us to reach v_1.

(f) $y = 8$ and $\hat{x} = 27$. With 8 pebbles on the arch, either 2, 3 or 4 pebbles arrive to v_5. If either 3 or 4 arrive, we can reach v_1 by sending all of the $27 + 3$ or $27 + 4$ pebbles across the path because the worst case puts all 27 pebbles on v_6, so 13 pebbles would arrive to v_5, plus either 3 or 4 gives us 16 or 17. From above, when $\hat{x} = 27$, at most 1 pebble will sit on v_4, so the worst case would again send 13 pebbles up the arch for 6 to arrive at u_4. The following graphs are the only configurations in which, with 8 pebbles on the arch, no more than 2 pebbles may arrive at v_5.

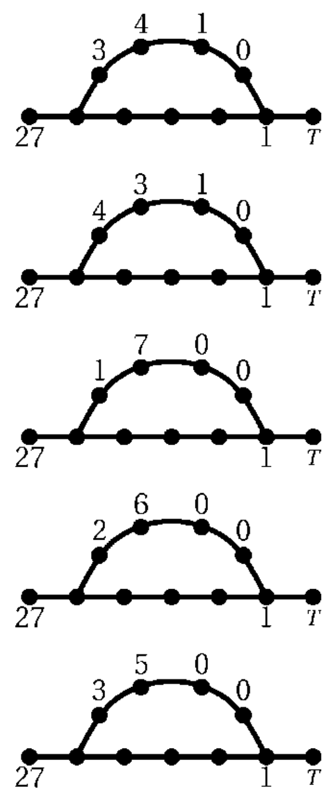


Figure 5.27: Five configurations in which $y = 8$ and 2 pebbles arrive at v_5.

In all of these cases, adding 6 pebbles to u_4 allows us to reach v_1.

(g) $y = 9$ and $\hat{x} = 26$. With 9 pebbles on the arch, either 3 or 4 pebbles arrive to v_5 from the arch. Then, with 26 pebbles on the path, the worst case occurs when all 26 pebbles are on v_6, so 13 pebbles arrive to v_5 plus the 3 or 4 from the arch. This means that we can always reach v_1 by sending all pebbles across the path.

(h) $y = 10$ and $\hat{x} = 25$. From above, when $\hat{x} = 25$, pebbles can only sit on vertices v_4, v_5, and v_6. If v_4 has at least 3 pebbles, we can reach v_1 by sending the remaining at most 22 pebbles across the path. If v_4 has 0, 1 or 2 pebbles, worst case is that 12 pebbles arrive to v_5, so 6 pebbles arrive to u_4. Then with $6+10$ pebbles on the arch, we can reach v_1.

(i) $y = 11$ and $\hat{x} = 24$. With 24 pebbles between v_3, v_4, v_5, and v_6, worst case is that 11 pebbles arrive to v_5, so 5 pebbles arrive to u_4. Then the arch has $11+5$ pebbles, so we can reach v_1.

(j) $y = 12$ and $\hat{x} = 23$. With 12 pebbles on the arch, all configurations that do not reach v_1 will reach v_5 either 5 or 6 times. Combining these with the 23 pebbles on the path, v_1 can always be reached by sending all pebbles across the path.

(k) $y = 13$ and $\hat{x} = 22$. With 22 pebbles on the path, the worst case scenario will reach v_5 10 times. This means at least 5 pebbles will reach the arch. Combining these ≥ 5 pebbles with the 13 pebbles on the arch, v_1 can always be reached.

(l) $y = 14$ and $\hat{x} = 21$. With 14 pebbles on the arch, the two configurations that do not reach v_1 will reach v_5 either 6 or 7 times. Combining these with the 21 pebbles on the path, v_1 can always be reached by sending all pebbles across the path.

(m) $y = 15$ and $\hat{x} = 20$. With 20 pebbles on the path, the worst case scenario will reach v_5 9 times. This means at least 4 pebbles will reach the arch. Combining these ≥ 4 pebbles with the 15 pebbles on the arch, v_1 can always be reached.

Any possible configuration of 67 pebbles on this graph reaches T. This means that 67 is the smallest possible number such that we can always reach T for any vertex T, i.e. 67 is the pebbling number of this graph. □

5.3 Conclusion

There is still much work that can be done towards finding the pebbling number of trampoline graphs. In this thesis, I have presented the lower bounds of the pebbling number in the general case, as well as specific values of the pebbling number for two particular cases.

I am currently exploring the general proof for the pebbling number of trampoline graph with my advisor, but we are in too early a stage for any of the results to be presented in this thesis.

List of Figures

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