POLYGONAL TILINGS OF THE PLANE

BY

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Polygonal Tilings of The Plane

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Tiling the Plane by Polygons

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INTRODUCTION

A ***tiling*** is a covering of the plane with non-overlapping figures that have no holes between them. For centuries, civilizations have incorporated both the complexity as well as aesthetic properties of tilings into their art and everyday lives. No matter where we look today, from religious to modern day buildings, to the typical household decorative arts, we can see some form of tiling being displayed. In this research, we will study tilings of the plane by various polygons. We want to determine whether, it is possible to tile the plane with a given set of specified figures. We will first look at basic properties of tilings as well as the properties of polygons.

Definition 1: A ***monohedral tiling*** is a tiling where identical polygons are used to cover the plane. For example, tilings in Figures 1, 2, 3 are monoheral, while the tiling in Figure 4 is not.

Definition 2: A ***regular tiling*** is a monohedral tiling of the plane by regular polygons. For example, see Figures 16 and 17.

Definition 3: A ***corner*** in any tiling is a corner of one of the polygons.

Definition 4: A ***vertex*** is a point where at least three tiles meet.

Notice from the Figure1 below that not every corner is a vertex.

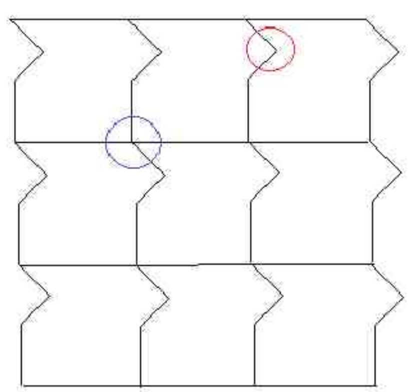


Figure 1. A corner(red) and a vertex(blue) in a tiling.

Definition 5: A tiling is ***edge-to-edge*** if two adjacent tiles have at most one edge in common and each edge belongs to exactly two tiles.

For example, tilings in Figures 1 and 2 are not edge-to-edge, while tilings in Figures 3 and 5 are edge-to-edge.

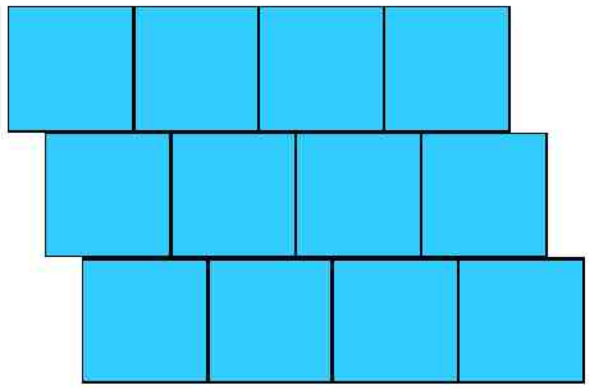


Figure 2. A tiling with squares that is not edge-to-edge.

Definition 6: A tiling is ***monogonal*** if at each vertex and corner, the edges form a figure that is congruent to that of any other vertex and corner.

For example, tilings in Figures 2, 3, 5 are monogonal, while tilings in Figures 1 and 4 are not.

Definition 7: A tiling is ***periodic*** if there exists a region in the tiling that can be translated in non-parallel directions in such a way that the entire tiling overlaps.

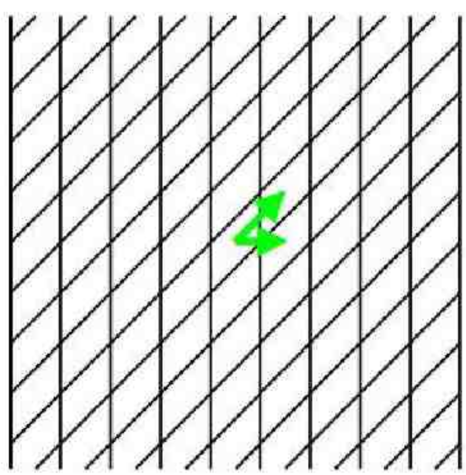


Figure 3. A periodic tiling.

Note that all tilings in Figures 1-5 are periodic, while tilings in Figures 38A, 38C, and 37B are not periodic.

Definition 8: A tiling is ***isogonal*** if each vertex and corner can be reflected, glided, rotated or translated onto any other vertex and corner in such a way that the entire tiling overlaps with itself.

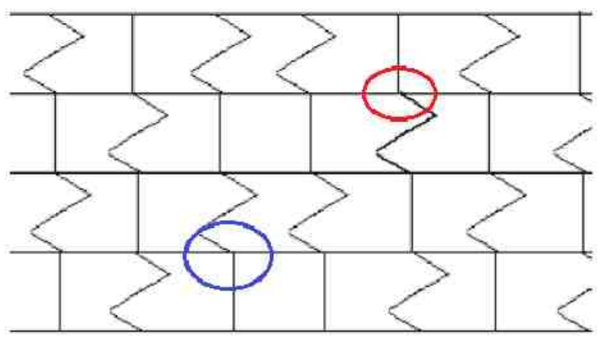


Figure 4. A non-isogonal tiling.

As an example, consider the tiling in Figure 4. Since more than one polygon was used to tile the plane, it is not a monohedral tiling. Consider the vertices circled in blue and red. Even though the figure formed at each vertex is identical, the vertex in blue is adjacent to a square, whereas the vertex in red does not. Hence we cannot reflect, glide, rotate or translate the vertex in red onto the vertex in blue. Therefore, the tiling is not isogonal.

Definition 9: A tiling is ***isotoxal*** if each edge can be reflected, glided, rotated or translated onto any other edge in such a way that the entire tiling overlaps with itself.

For example, the tilings in Figures 16 & 17 are isotoxal, while the tiling in Figure 5 below is not.

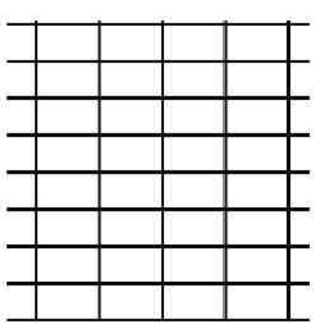


Figure 5. A non-isotoxal tiling. We cannot reflect, glide, rotate or translate the shorter edges onto the longer ones.

POLYGONS

In this chapter, we will study basic properties of various polygonal tiles.

Definition 10: A ***polygon*** is a two dimensional shape that is comprised of non-intersecting segments of straight lines connected in a cyclic way to form the boundary of a closed shape.

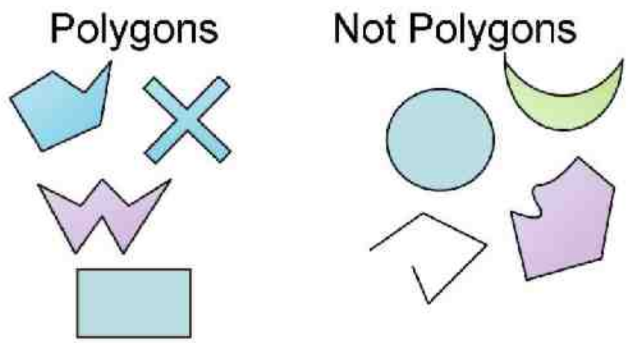


Figure 6. Polygons and non-polygons.

A polygon with one or more interior angles greater than 180^c is called a ***concave*** polygon, while the interior angles of a **convex** polygon are all less than 180^c (Figure 7). Since we will study several tilings by concave polygons, we distinguish types of concave polygons by definitions 11-13 below.

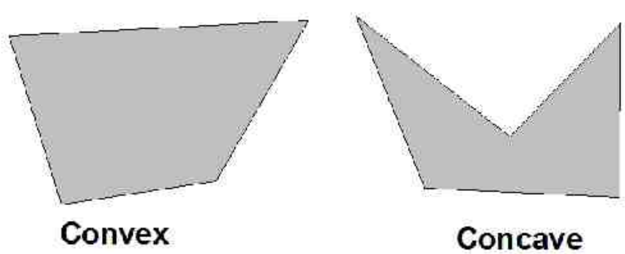


Figure 7. The concave polygon has at least one interior angle greater than 180°.

Definition 11: A ***cave*** in a concave polygon is a cavity formed by exactly two edges that meet to form a vertex with interior angle greater than 180°.

Definition 12: A ***bay*** is a cavity in a concave polygon formed by two or more consecutive vertices with interior angles greater than $180°$. In other words, a bay is a cavity formed by two or more consecutive caves.

Definition 13: A ***simple cave*** is a cave that does not belong to a bay.

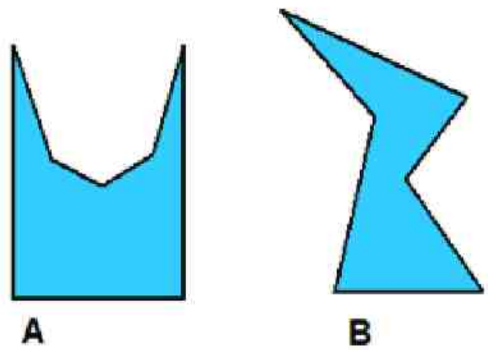


Figure 8. A concave polygon with a bay A), and a concave polygon with two simple caves B).

Lemma 1: For any integer $n > 3$, there exists a non-convex polygon with n edges that has c caves, where $c \in \left\{1\ 2, \dots, \frac{n}{2}\right\}$ for n even, and $c \in \left\{1\ 2, \dots, \frac{n-1}{2}\right\}$ for n odd.

Proof: There are two cases to consider.

Case 1: $n > 3$ is even.

Notice that every cave is made up of exactly two edges. Since any two simple caves cannot share an edge and we have n edges total, it is possible combinatorially to have a concave polygon with $\frac{n}{2}$ caves or less. Constructing a polygon with $\frac{n}{2}$ caves is not very difficult. For example, a star-like tile carries the maximum number of caves, see Figure 9A.

Case 2: n is odd.

Since each cave is formed by two edges that cannot be shared with other caves, and because n is odd, combinatorially it is possible to have a concave polygon with $\frac{n-1}{2}$ caves or less.

For example, a star-like tile with an additional "base" edge that has the maximum number of caves looks similar to the tile in Figure 9B.

Q.E.D.

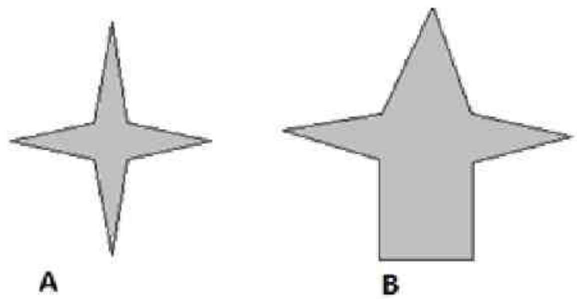


Figure 9. Concave tiles with eight A), and nine B) edges that have the maximum number of caves.

Lemma 2: For $n = 5$ or 6, there exists a polygon with n edges and one bay. For any integer $n > 6$, there exists a polygon with n edges, that has b bays where

$$1 \leq b \leq \left\lfloor \frac{n}{3} \right\rfloor = k .$$

Proof: Note that each bay is made up of at least 3 edges and a polygon with a bay, must have at least five edges. In Figure 10 below, we can see a 5-gon and a 6-gon with the maximum number of bays.

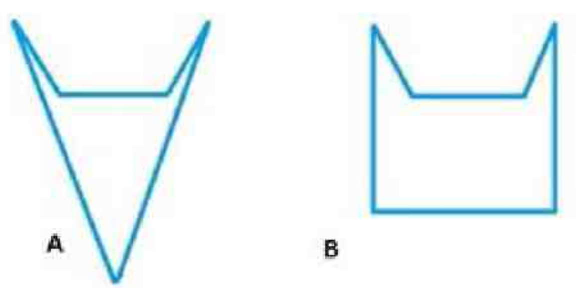


Figure 10. Concave tiles with five A), and six B) edges that have the maximum number of bays.

For $n > 6$, we can have at most $\left\lfloor \frac{n}{3} \right\rfloor = k$ bays. In Figure 11 below, we display tiles with 1 bay and 4 bays in the case that $n = 12$.

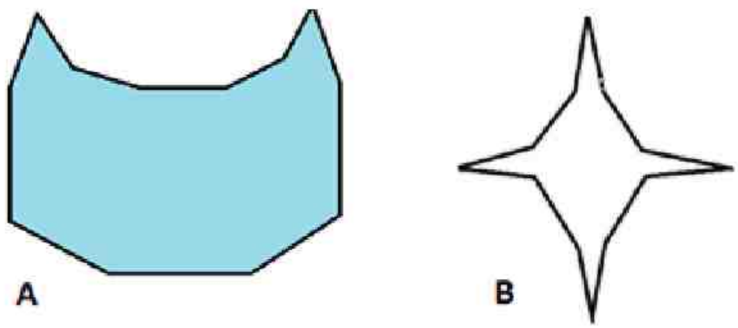


Figure 11. Concave 12-gon with A) one bay and B) four bays.

Q.E.D.

Definition 14: A polygon is called ***regular*** if it has congruent sides as well as all congruent interior angles.

The names for these polygons come from Greek and they express the number of angles in a given polygon. By convention dating back to the ancient Babylonians, these angles are measured in degrees.

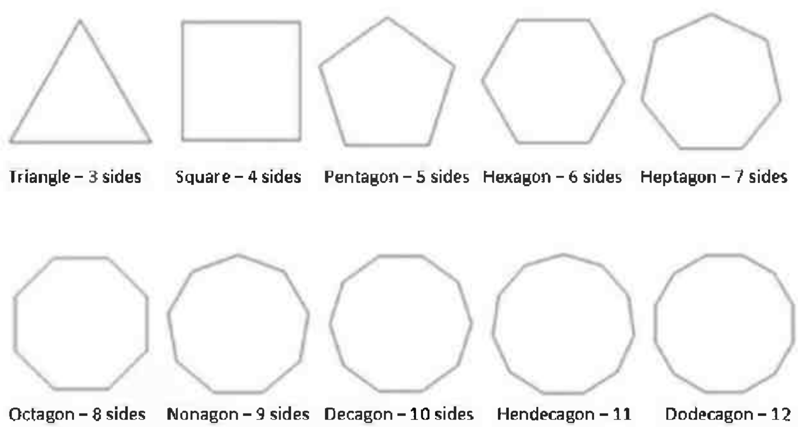


Figure 12. Examples of Regular Polygons.

Lemma 3: The sum of the interior angles in a polygon of n sides $S(n) = (n-2)180°$.

Proof by Induction:

Base Case $(n = 3)$ – For a triangle we have $S(3) = (3-2)\ 180° = 180°$.

Assume the lemma is valid for a given number k, i.e. $S(k) = (k-2)\ 180°$.

We want to show that the lemma is valid for $k+1$, i.e. $S(k+1) = (k-1) \cdot 180°$.

We will label the vertices of our k-sided polygon in a clockwise manner, $v_1, v_2, \dots\dots, v_k$. Note that by adding a vertex to the k-sided polygon, we have added a triangle to make the $(k+1)$-sided polygon. In other words, the $(k+1)$-sided polygon is the union of our original k-sided polygon $v_1, v_2, \dots\dots, v_k$ and triangle v_1, v_k, v_{k+1} (Figure 13). It is a well known fact that any n-gon can be partitioned into $n-2$ non-overlapping triangles using diagonals only [Gr]. For this reason, if we instead added a vertex to the interior of the k-sided polygon, the results would not change and we have again added a triangle to make the $(k+1)$-sided polygon. Our k-sided polygon satisfies our hypothesis for $S(k)$ and since the sum of the interior angles of a triangle is $180°$, hence we have the following :

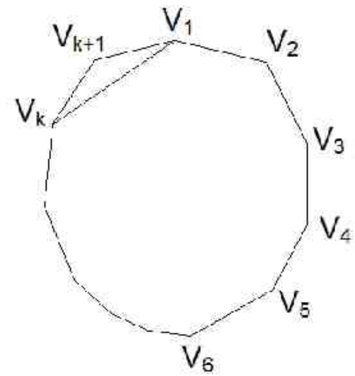


Figure 13.

$$S(k+1) = S(k) + 180°$$

$$= (k-2) \cdot 180° + 180°$$

$$= ((k-2)+1)\ 180°$$

$$= (k-1)\ 180°.$$

Q.E.D.

Since the interior angles in a regular polygon are all equal, taking the sum in Lemma 3 for an n-sided polygon and dividing it by n will give us the measure for the interior angles of that particular n-sided polygon.

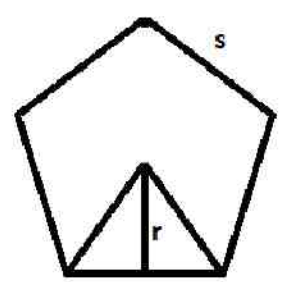


Figure 14.

Lemma 4: The area of a regular polygon with n sides is $A(n) = \frac{n}{2}(sr)$ where n is the number of sides, s is the side length, and r is the radius of the inscribed circle.

Proof: Each n-sided polygon can be partitioned into n congruent triangles by drawing a line segment from each vertex to the center of the polygon. The area of each triangle in the partition is $\frac{sr}{2}$ and since we have n triangles, the area of the entire n-sided polygon is $A(n) = \frac{nsr}{2}$ (Figure 14).

Q.E.D.

<u>Lemma 5:</u> The number of diagonals in an n-sided convex polygon is $D(n) = \frac{n(n-3)}{2}$.

Proof by Induction:

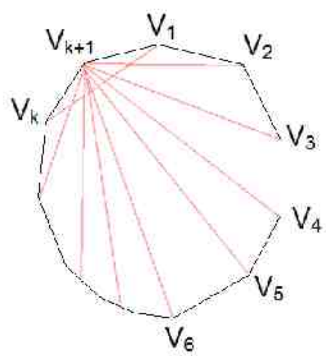


Figure 15.

Base Case $(n = 4)$ – By inspection, a quadrilateral has $D(4) = \frac{4(4-3)}{2} = 2$ diagonals Assume $D(k) = \frac{k(k-3)}{2}$ is true for $k \geq 2$. We want to show that the formula holds for $D(k + 1)$. Note that if we add a vertex to a k-sided polygon, we have a total of k lines connecting vertices $v_1, \ldots, v_k$ to vertex v_{k+1}. Of these k lines, we subtract 2 since they are not diagonals but new sides. Also, one side of k is now a diagonal as a result of adding a vertex (Figure 15).

Hence we have the following calculation:

$$D(k + 1) = D(k) + k - 2 + 1$$

$$= \frac{k(k-3)}{2} + k - 1$$

$$= \frac{k^2 - 3k + 2k - 2}{2}$$

$$= \frac{k^2 - k - 2}{2}$$

$$= \frac{(k+1)(k-2)}{2}.$$

Q.E.D.

Example: Here we analyze the nonagon, which has nine sides as well as nine vertices. For a nonagon, the sum of its interior angles is $7 \cdot 180° = 1260°$ which implies that the interior angles for a regular nonagon measure $\left(\frac{1260°}{9}\right) = 140°$. The nonagon has $\frac{9(6)}{2} = 27$ diagonals.

There exists regular polygons that can be constructed using a straight-edge and a compass. However, many regular polygons cannot be constructed in this manner. The problem of deciding whether a regular polygon can be constructed was first considered by ancient Greek mathematicians. However, it was not until the 19th century, when Gauss developed a formula that decides which regular polygon can be constructed using a straight-edge and a compass.

Theorem 1 (Gauss): Any constructible regular polygons must satisfy the formula $n = 2^m p_1 p_2 \ldots\ldots p_k$, where $m \in N$ including 0, and $p_i{}'s$ are different Fermat's primes of the form $2^{2^t} + 1$.

Proof: See reference [Gr].

Note that if the construction of a particular regular n-sided polygon is possible, then construction of the regular $2n$-sided polygon is also possible by bisecting the sides of the polygon inscribed in a circle. For example, Gauss's formula tells us that it is not possible to construct the regular nonagon with a straight edge and compass. Therefore, to draw a regular nonagon, we must rely on methods that involve measurements of sides or angles which can only give us a close approximation of the figure.

Every regular polygon exhibits high degree of symmetry, each having both rotation and reflection symmetries of a dihedral rosette pattern [Gr]. For any n-gon, the symmetry group is denoted by D_n and it has n reflectional symmetries and has n rotations generated by a rotation with angle $\frac{360°}{n}$.

TILING BY REGULAR POLYGONS

We first investigate simple tilings and analyze their basic forms and properties. The most common regular monohedral tiling that is edge-to-edge, seen in the majority of tiled floorings, is given by a square (regular 4-gon). Since each square has four right interior angles, we can place four identical squares around one vertex and they will fit perfectly, since the sum of the angles at that corner is $360°$. Hence, by repeating this process infinitely many times for each corner, we will tile the plane (Figure 16).

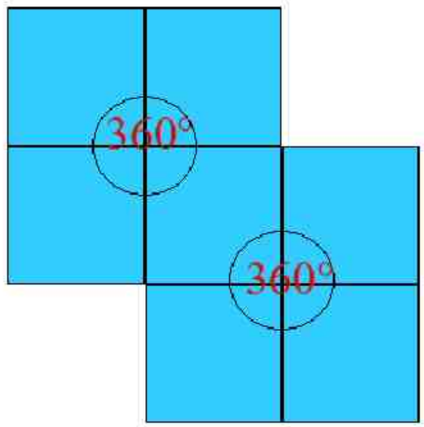


Figure 16. A monohedral edge-to-edge tiling with squares.

Using a similar argument, we can show that identical regular triangles as well as a regular hexagons will also tile the plane. Since the interior angles of a regular triangle measure $60°$, placing six triangles at a vertex gives us an edge-to-edge tiling of the plane. Similarly, we will need three identical regular hexagons at each vertex in order to produce a hexagonal tiling, as $3(120°) = 360°$.

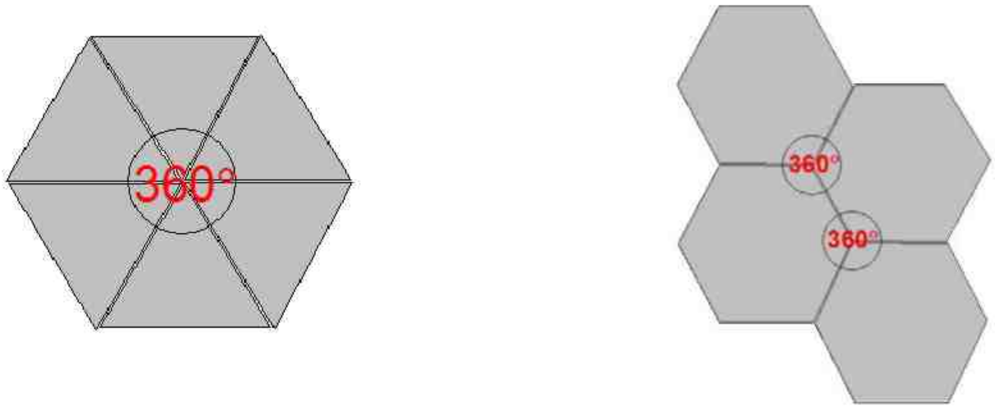


Figure 17. Monohedral edge-to-edge tiling with regular triangles and hexagons.

Notice that the regular tiling of the plane by triangles, squares, and hexagons are monogonal, edge-to-edge, and isogonal (the figure formed at the vertices are identical and can be reflected, glided, rotated or translated onto one another so that the tiling overlaps with itself). In addition, these tilings are also periodic as well as isotoxal.

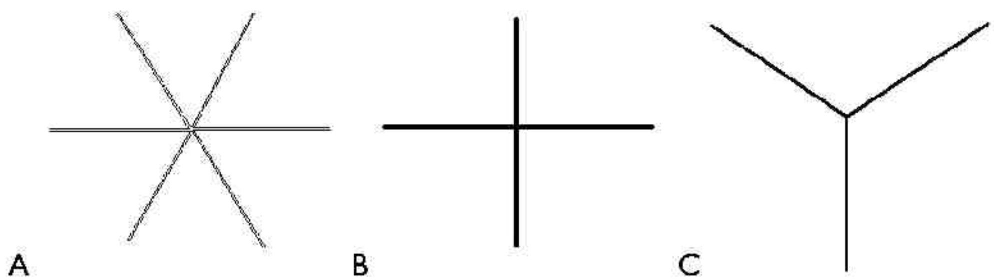


Figure 18. The congruent figures formed at the vertices in a A) triangular, B) square, and C) hexagonal tilings.

To study square tilings, we could use the following 2-dimensional case of Keller's conjecture.

Keller's Conjecture: A tiling of an n-dimensional space by n-dimensional hypercubes of equal size will produce an arrangement such that at least two hypercubes have an $n-1$-dimensional side in common.

Note that, a 2-dimensional hypercube is a regular square, while a 1-dimensional hypercube is a line segment. Keller's conjecture states that a tiling of the plane by congruent squares must have at least two squares with an edge in common, or some two squares must meet edge-to-edge.

Proof: The proof for the 2-dimensional case is elementary and we provide it here. For a proof in general case see [W].

Consider the 2-dimensional hypercube (square). In any tiling, we know that there cannot be overlapping of the tiles and also no gaps. Therefore, the first two tiles in the tiling must meet either at some angle or at some edge. The two cases are analyzed below.

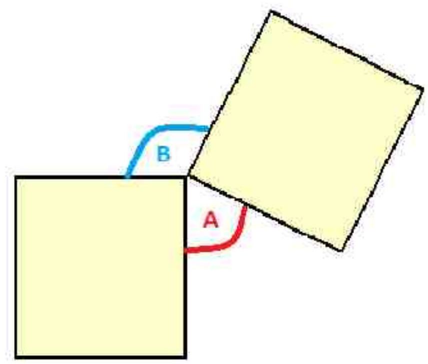


Figure 19.

Case 1- (The tiles meet at a corner seen in Figure 19 at the right.): If the first two tiles meet at a corner and forms an angle A that is exactly 90°, then we can see that then angle B must also be 90°. Therefore, in this case to cover the plane without overlaps, we place the next tiles at angles A and B, i.e. edge-to-edge.

If the first two tiles meet at a corner and forms an angle A that is less than 90°, then we can see from the picture that angle B must be greater than 90°. Hence, placing a tile at B is will not cover the plane and it is not possible to fit a tile at A. Therefore, to tile the plane angle A as well as B must be 90° and the tiling generated in Case 1 will be that of Figure 16. Hence there are at least two tiles that meet edge-to-edge.

Case 2 - (The tiles meet in a way such that that their edges touch seen in Figure 20.): If the first two tiles meet at an edge, we can see that both angles A and B which are formed must be equal to 90°. Therefore, in order to continue the tiling pattern such that there are no gaps, we must place a tile at corner A and corner B. The tiling generated in Case 2 will be that of Figure 2 and we can see that at least two tiles meet each other edge-to-edge.

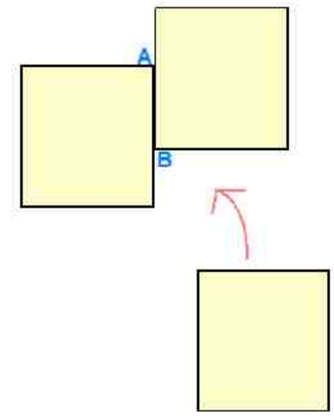


Figure 20.

Q.E.D.

Remark: The square tiling generated from case one of the proof above is edge-to-edge, while that in case two is not edge-to-edge. In Figure 21 below, we can see that Keller's conjecture does not extend to non-hypercubes.

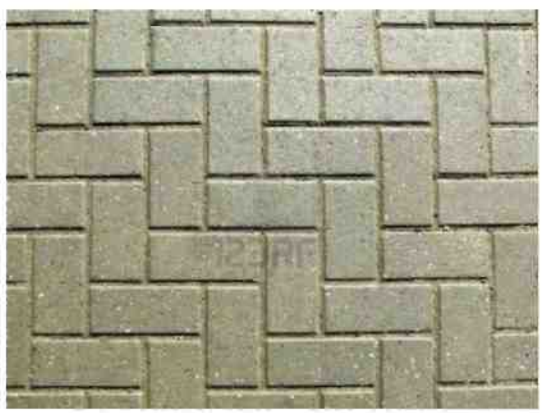


Figure 21. A tiling where no two tiles have an edge in common.

Theorem 2: The triangle, square, and hexagon are the only regular polygons that will tile the plane.

Proof: Let $x \in N$ be the number of tiles at each vertex and $n \in N$ be the number of sides of a regular polygon.

Hence we have at each vertex:

$$\frac{x(n-2)180^\circ}{n} = 360^\circ$$

Simplifying we get:

$x(n-2) = 2n$

Hence, $x = \frac{2n}{n-2}$

Since x is a positive integer, $n \in \{3, 4, 6\}$.

Q.E.D.

Corollary: There does not exist a tiling of the plane by regular pentagons and regular n-sided polygons for $n > 6$.

We can see from Figure 22 below and angle calculations that an edge-to-edge tiling of the plane with regular pentagons is not possible.

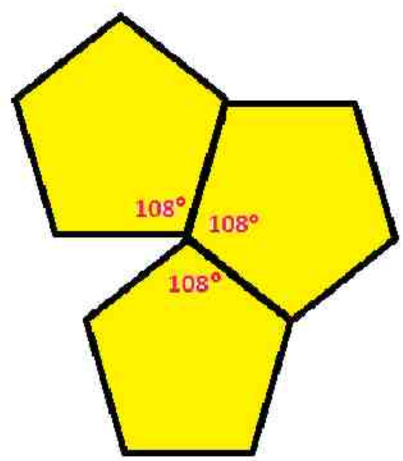


Figure 22. Tiling with regular pentagons is not possible since we do not have a perfect fitting at the vertices.

We have shown that it is not possible to tile the plane with many regular n-sided polygons. Therefore, it is natural to study monohedral tilings by other polygons.

TILING BY IRREGULAR POLYGONS

In this section, we will investigate monohedral tilings of the plane by irregular polygons.

Definition: An **irregular polygon** is a polygon whose sides or interior angles are not all equal.

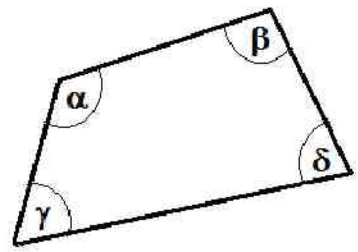


Figure 23. An irregular polygon.

Theorem 3: A plane can be tiled by using identical triangular tiles of any shape.

Proof: We distinguish three classes of triangles, equilateral (regular), isosceles, and scalene. We have already shown that an equilateral triangle will tile the plane in the last section. Now, we must show that an isosceles and scalene triangle will also tile the plane.

First, consider the scalene triangle seen in Figure 24 below with no congruent sides and no equal interior angles.

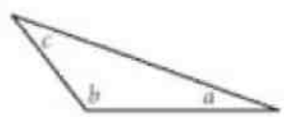


Figure 24. A scalene triangle.

Notice that if we place the tile in the following manner, we can tile the plane.

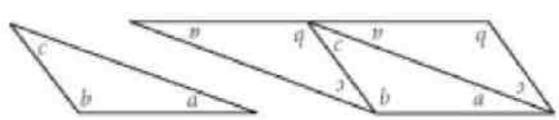


Figure 25. Construction of a tiling by scalene triangles.

Now, consider the isosceles triangles seen below.

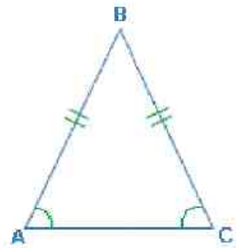


Figure 26. An isosceles triangle.

A tiling of the plane by isosceles triangle can be seen below.

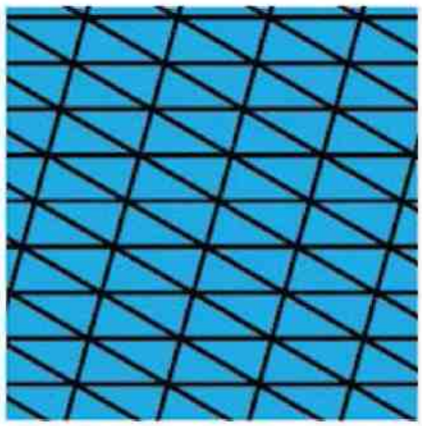


Figure 27. A tiling by isosceles triangles.

Q.E.D.

Theorem 4: A plane can be tiled using identical quadrilateral tiles of any shape.

Proof: We have seen that a regular quadrilateral (square) will tile the plane. Now we analyze the remaining cases.

First, consider the non-regular convex quadrilateral.

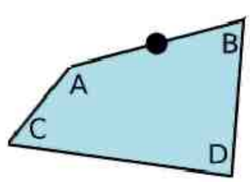


Figure 28. A non-regular convex quadrilateral.

We can construct a tiling by rotating the tile by 180° about the midpoint on one of its sides and repeating the process with the other sides.

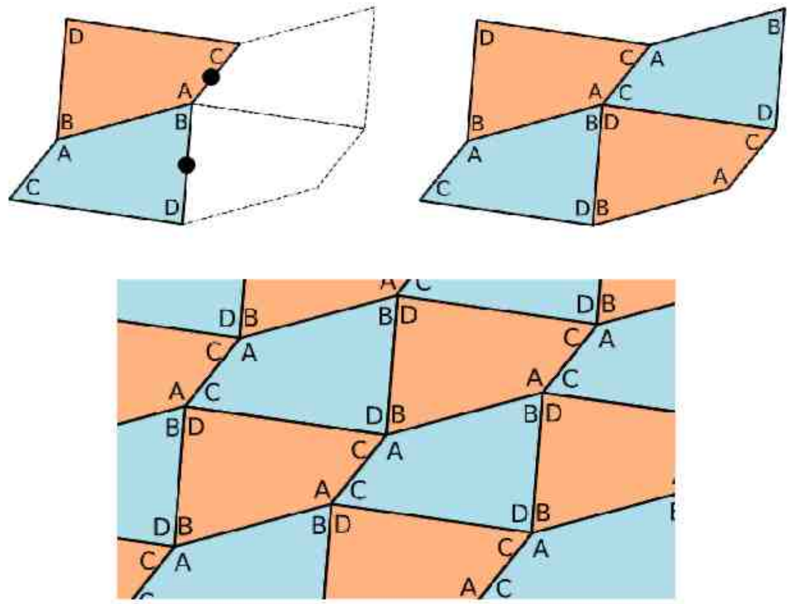


Figure 29. A tiling with non-regular convex quadrilaterals.

We can see that the vertices in our tiling are made up of exactly the angles from our original quadrilateral. Since the angles in a quadrilateral add up to 360°, our tiling has no gaps or overlaps.

For a concave quadrilateral, we can repeat the same procedure and we will construct a tiling similar to the one in Figure 30 below.

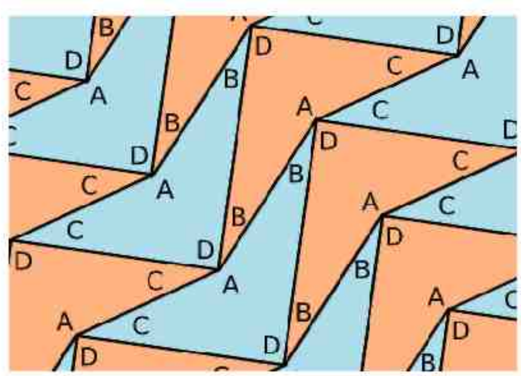


Figure 30. A tiling with concave quadrilaterals.

Q.E.D.

In the previous section, we have shown that a regular pentagon does not tile the plane.

Theorem 5: There exists tilings of the plane by irregular pentagons. However, not every irregular pentagon tiles the plane.

Proof: See Figure 31 for examples. These are the only known monohedral convex pentagonal tilings.

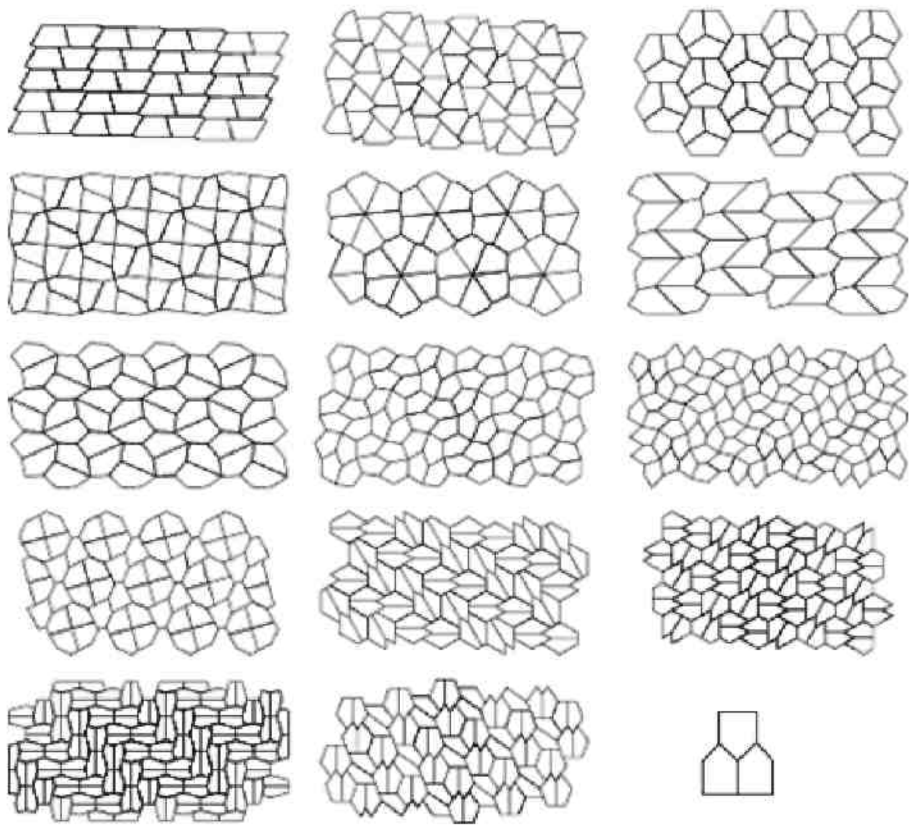


Figure 31. Fifteen classes of known tilings with convex pentagons. Even now, it is still a mystery whether or not more solutions exist. [W2]

We know that regular hexagons can tile the plane, but what about irregular hexagons? This problem was solved by Karl Reinhardt and published in his doctoral thesis in 1918.

Theorem 6 (Reinhardt's): There exist infinitely many tilings with convex irregular hexagons. These tilings can be divided into three classes (see Figure 32).

Proof: See [R].

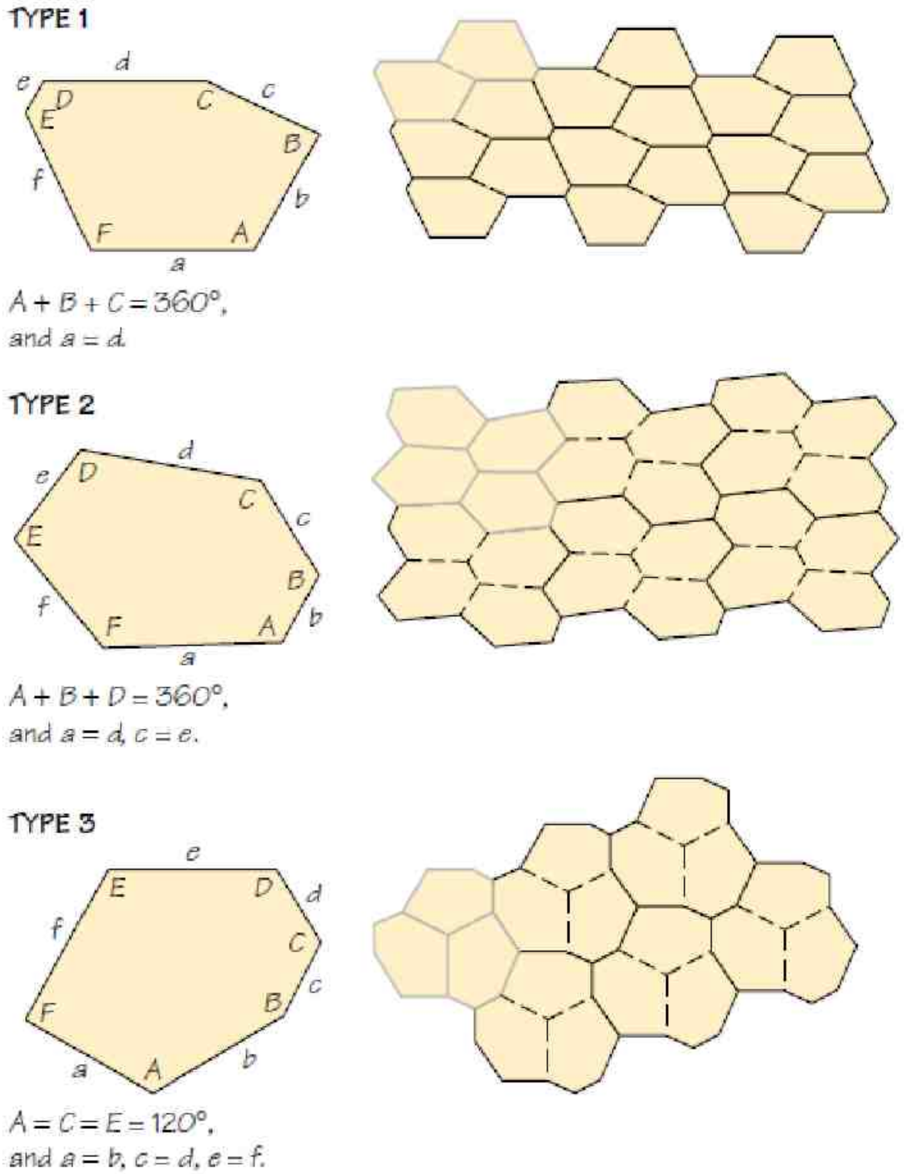


Figure 32. The only three classes of tilings with convex hexagons that exist. [F]

It turns out that hexagons play a major role in the study of monohedral tiling, as we have the following conjecture.

Honeycomb Conjecture: Any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal honeycomb tiling.

Proof: This was proven by Thomas Hales in 1999. See [H].

To study tilings for $n > 6$, we prove the following theorem.

Theorem 7: There exists no tilings by identical convex polygons that have k edges for $k > 6$.

Proof: (By contradiction):

Assume that there exists a tiling by identical convex polygons that have k-edges where $k > 6$. Note that Euler's characteristic is a topological invariant, a number that describes a topological space's structure regardless of the way it is bent. Euler's characteristic, denoted by χ, is given by the formula $\chi = F - E + V$ where F, E, and V are the number of faces, edges, and vertices [Gr]. Consider the figure below that shows a construction of a torus by gluing the edges and vertices respectively.

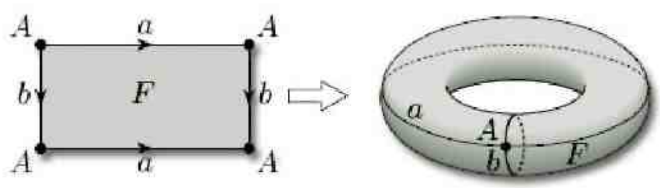


Figure 33. Constructing a torus from a rectangle. [OU]

Topologically, a torus has 1 face, $\frac{4}{2} = 2$ edges, and $\frac{4}{4} = 1$ vertex, and its Euler characteristic χ is

$$\chi = 1 - \frac{4}{2} + \frac{4}{4} = 0.$$

We will extend this gluing procedure done on a rectangle to a bounded region of our tiling by identical convex polygon with $k > 6$ edges. Let us choose a bounded region R of this tiling and consider only the edges that bound the region. It is always possible to select a region that has an even number of boundary edges and perform a similar gluing procedure as illustrated for the torus. To make sure that all the edges and vertices glue together nicely, we will subdivide the perimeter of the region R into 4 parts described in Figure 34.

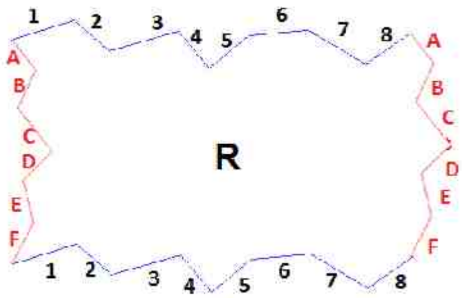


Figure 34. A selected region of our tilling used to form a torus.

We will first glue the blue edges with the same label, and then glue the red edges with the same label. By performing this gluing procedure on the topological region R, we have formed a torus; hence we expect $\chi = 0$. Now we calculate the number of faces, edges, and vertices of this torus, formed from our selected tiling region, and figure out Euler characteristic. Let n be the number of faces or tiles in our selected bounded region R of the tiling, and k be the number of edges and vertices of each polygon or tile. Therefore, we have a total of nk number of edges and vertices from the untiled polygons. Since our tiling is edge-to-edge, every two edges glued together count as one in the tiling, hence we have $\frac{nk}{2}$ number of edges in our torus. Note that our polygons are convex so every interior angle is less than $180°$. Hence, at every vertex in our tiling, there **must** be at least three polygons meeting. Therefore, we have at most $\frac{nk}{3}$ number of vertices in our torus.

Hence using Euler's formula, we obtain the following inequality:

$$0 \leq n - \frac{nk}{2} + \frac{nk}{3}$$

$$= n - \frac{3nk}{6} + \frac{2nk}{6}$$

$$= n - \frac{nk}{6}$$

$$= n\left(\frac{6-k}{6}\right).$$

However, for every n, when $k > 6$, the equation yields a negative number, a contradiction. Therefore, there are no monohedral tilings by convex polygons with more than six sides.

Q.E.D.

Remark: In the proof of Theorem 7, we did not use the fact that the tiles in the tiling are identical. Therefore, we can generalize our theorem as follows.

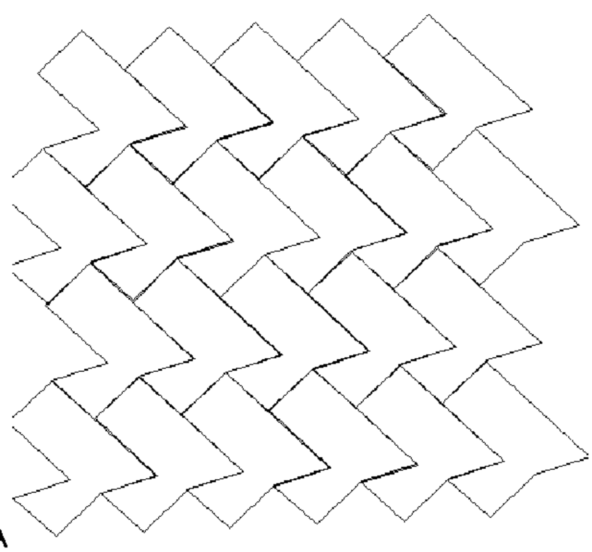
Theorem 8: There exist no tilings with any convex polygons that have more than six edges.

Proof: The proof follows from the arguments in Theorem 7 by slight modification of calculations of expected Euler characteristics.

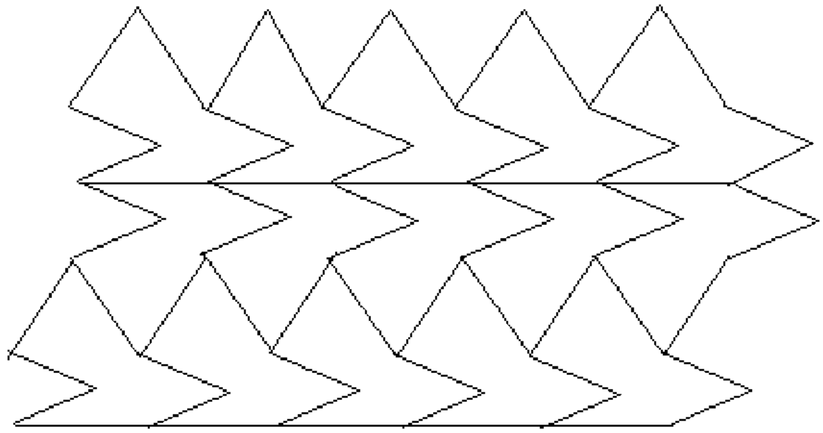
TILING BY CONCAVE POLYGONS

To illustrate the issues related to non-convex polygons, we will start by constructing several concave tilings and analyzing them.

Example I. Concave Heptagonal Tilings, $n = 7$.



A



B

Figure 35. Heptagonal tiling.

In Figure 35a), we have monohedral tiling of the plane by non-convex heptagons. The tiling is not edge-to-edge, since the longest edge belongs to more than two tiles. The tiling is not isogonal, as there are several types of vertices in the tiling. Also, since the sides of the heptagonal tile all have different length, the tiling cannot be isotoxal. In Figure 35b), the tiling is again not edge-to-edge, since there are adjacent tiles with more than one edge in common. Notice that the tiling also has more than one type of vertices, hence it is not isogonal. We also can see that the tiling has edges of different length and therefore the tiling cannot be isotoxal. However, these polygonal tiles seem quite interesting and have other properties that decide about their ability to tile the plane.

Example 2. Concave Octagonal Tiling, $n = 8$.

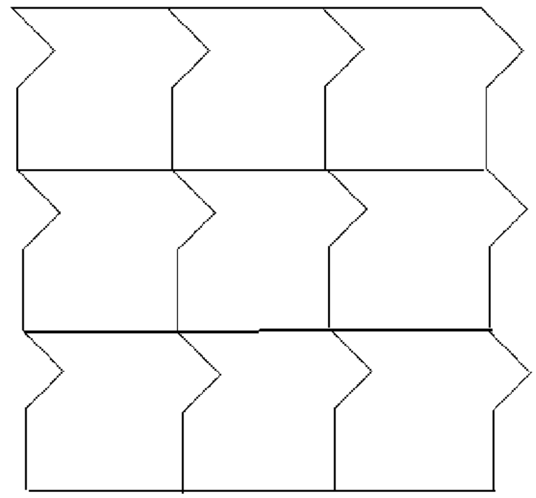
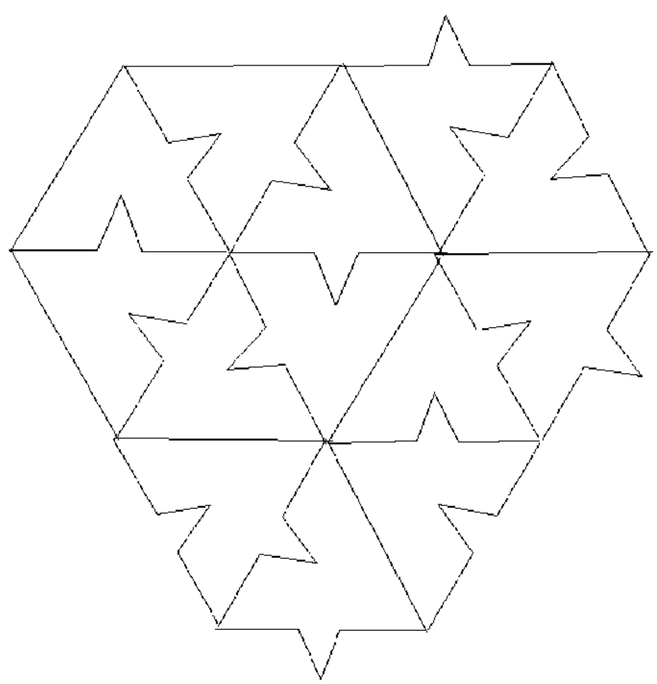


Figure 36. Octagonal Tiling.

The monohedral tiling of the plane with non-convex octagonal tiles above is periodic, not monogonal, not isogonal, not isotoxal, and not edge-to-edge.

Example 3. Concave Nonagonal Tilings, $n = 9$.



A)

B)

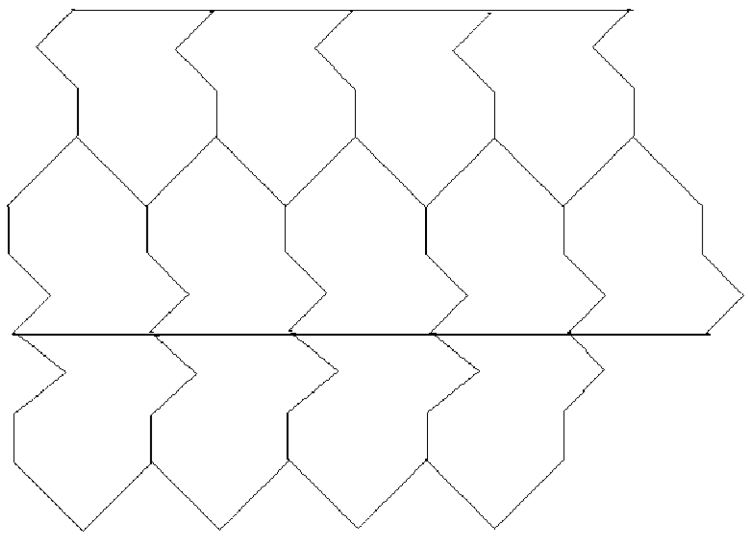
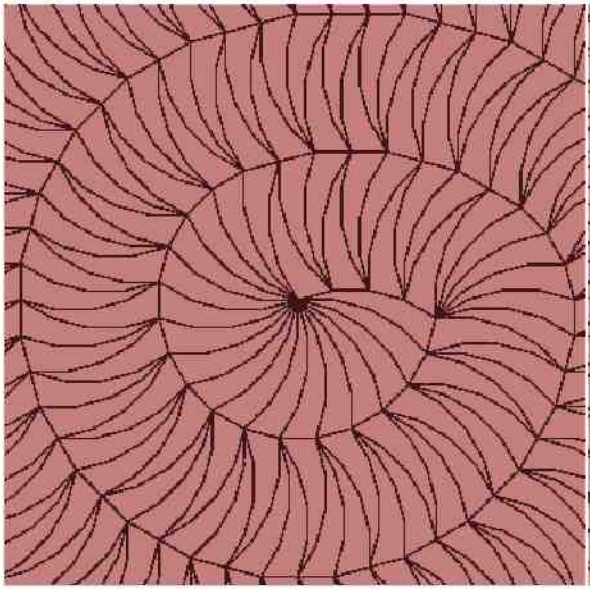
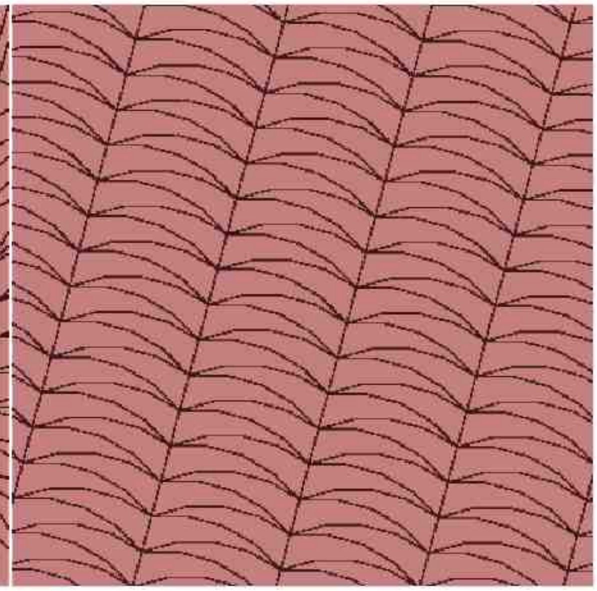


Figure 37. Nonagonal Tilings.

In Figures 37A and 38B, the monohedral tilings are not edge-to-edge, not monogonal, not isogonal, and not isotoxal. The tiling in Figure 37A is periodic while the tiling in Figure 37B is not periodic.



A



B

C

Figure 38. Variations of tilings of the plane using Voderberg's tile (nine sided).

In Figures 38A-C, the tilings are not edge-to-edge, not monogonal, and not isogonal. The tilings in Figures 38A and 38C are not periodic, since we cannot translate them in such a way that the entire tiling overlaps.

So far, we have shown that it is impossible to tile the plane with identical or non-identical convex tiles that have more than six edges. Our examples above in Figures 35-38 give tilings done with identical concave polygons with more than six edges. Hence, concave polygons give a large class of tilings that have interesting properties. We will start by proving several theorems. The table below will serve us as reference in the proofs for Theorems 9 ,12, and 13.

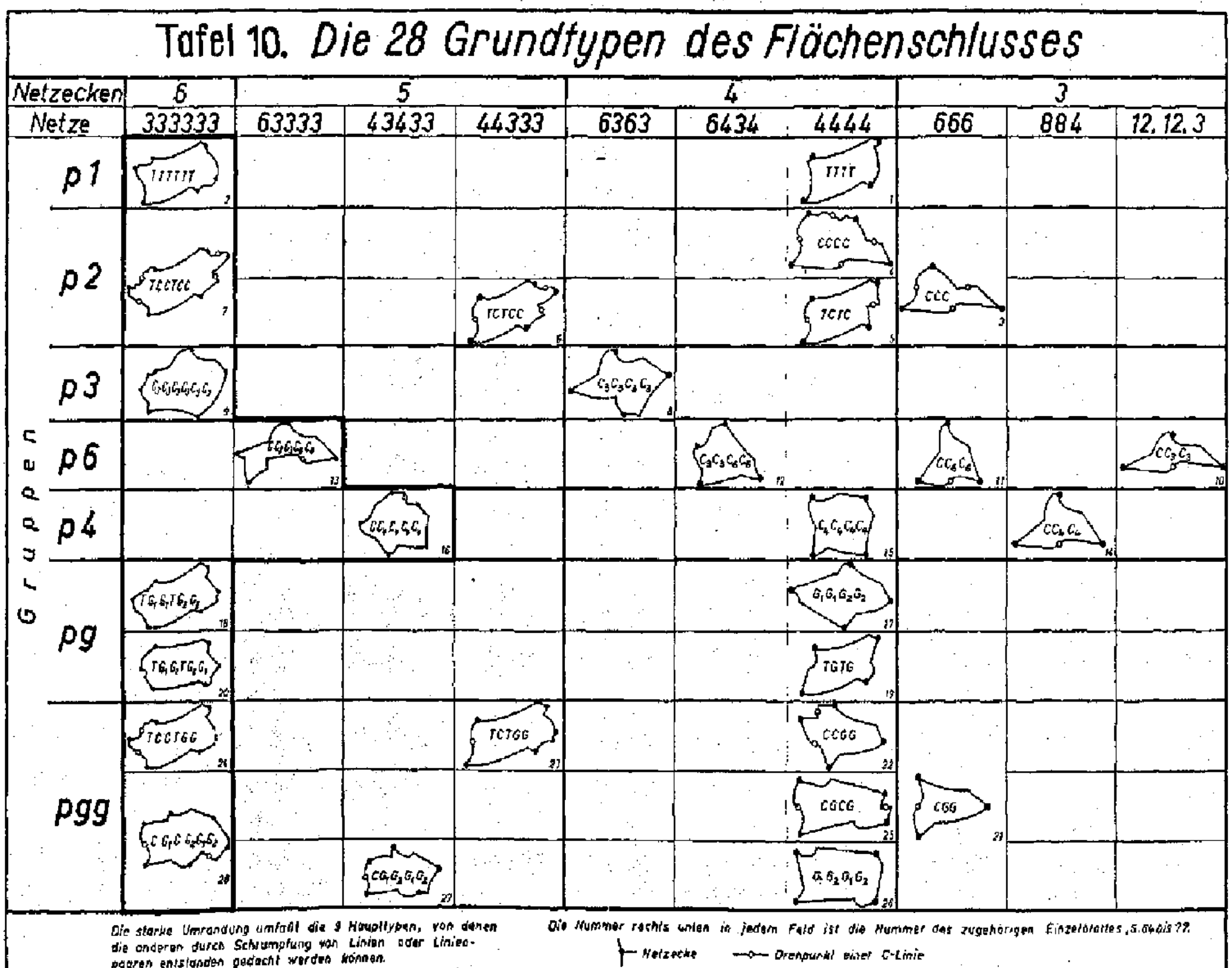


Figure 39. Heesch's table of the 28 asymmetric tiles. [HK]

Theorem 9: For every integer $n > 3$, there exists a monohedral tiling of the plane with identical concave polygons that have n edges and only simple caves.

Proof: By construction, we will show that it is possible to generate monohedral tilings by concave n-gons for any $n > 3$. First, consider a tiling given by triangles. We can modify the edges of this tiling in a certain way such that a monohedral tiling by different concave polygons is generated. In Figure 40, a tiling given by triangles is modified by substituting each edge around a given vertex with two edges propagated by a 60° rotation. This procedure can be continued with adjacent regular hexagons (that tiles the plane) and generates a monohedral tiling by concave polygons that have five edges each.

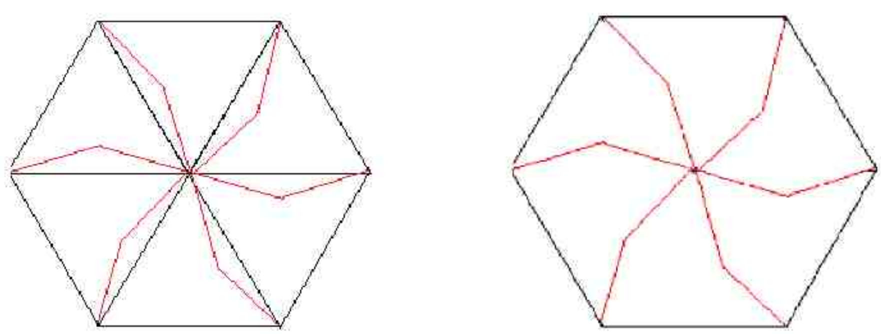


Figure 40. Construction of a monohedral tiling by concave 5-gons.

Our method is quite simple and is based on Heesch's classification of tiles [Gr]. If we look at the newly generated concave five sided polygon as a transformation of the triangle, then it has Heesch type of CC_3C_3. We can repeat this method of construction on the tiling generated in Figure 40 above by substituting one of the edges of the cave with two edges. However, to preserve the fact that our polygons have only simple caves, we substitute in the two edges in an opposite orientation as we did in the previous step in Figure 40. This procedure can be continued with the adjacent hexagons and generates a monohedral tiling by concave polygons with seven edges and only simple cave (see Figure 41).

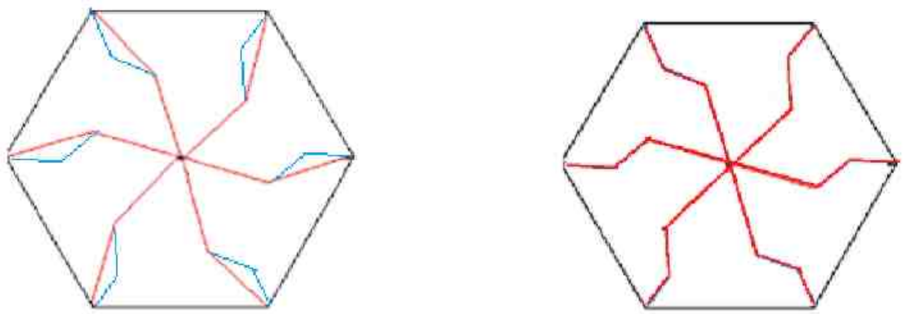


Figure 41. Construction of a monohedral tiling by concave 7-gons with only simple caves.

This newly generated concave 7-gon is a transformation of the triangle, and it has again a Heesch type of CC_3C_3. To summarize the construction method, we are taking one of the edges forming a cave and substituting it with two edges in alternating orientation (the current substitution of edges must have opposite orientation as the previous substitution). Iterating this construction, we can generate all the monohedral tilings by concave polygons with odd number of edges.

For the case when $n > 4$ and even, we start with the tiling by squares. Application of the of the previous method of construction on a square tiling allows us to generate, all the monohedral tilings with concave polygons with an even number of edges. In Figure 42 below, we show the construction of a monohedral tiling by concave 6-gon with only a simple cave.

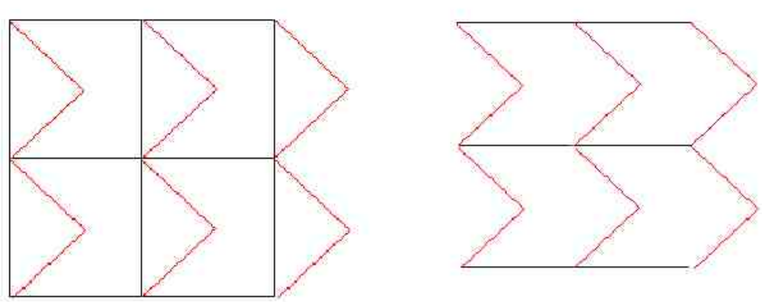


Figure 42. Construction of a monohedral tiling by concave 6-gons with only simple cave.

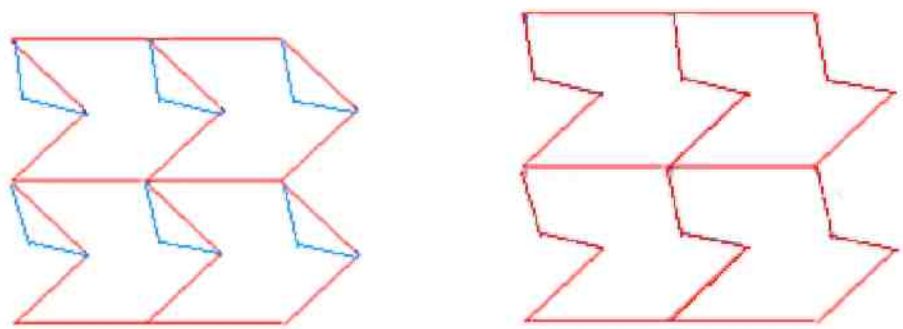


Figure 43. Construction of a monohedral tiling by concave 8-gons with only simple caves.

In Figure 43, we applied the construction method to the tiling by concave 6-gons in Figure 42 to generate a tiling by concave 8-gons with only simple cave. Again, this 8-gon is generated from a square and has Heesch type TTTT. As we can see that this method has a fractal pattern, since we are substituting an edge with two scaled down edges and iterating the process several times. In our construction, we start with either a triangular or a square tiling as our base to generate more complicated monohedral tiling with larger number of edges. Hence, we can consider the new lines as designs on our base tilings (similar to Escher's designs). Note that, our generated concave polygon with $n > 3$ edges will always be Heesch's type TTTT (n-even) or CC_3C_3 (n-odd).

Q.E.D.

Remark: There are other tilings by identical concave polygons that can be constructed using different methods (Figures 35-38).

TILING BY SYMMETRICAL POLYGONS

In this section, we will focus our attention on tilings of the plane by a more specific type of irregular polygons, those that have at least one line of symmetry.

Theorem 10: For every integer $n > 3$, there exists a monohedral tiling of the plane by concave polygons that have n-edges with only simple caves and at least one symmetry line.

Proof: We start by constructing symmetric tiles for small n. Figure 44 shows examples of tiles which interest us.

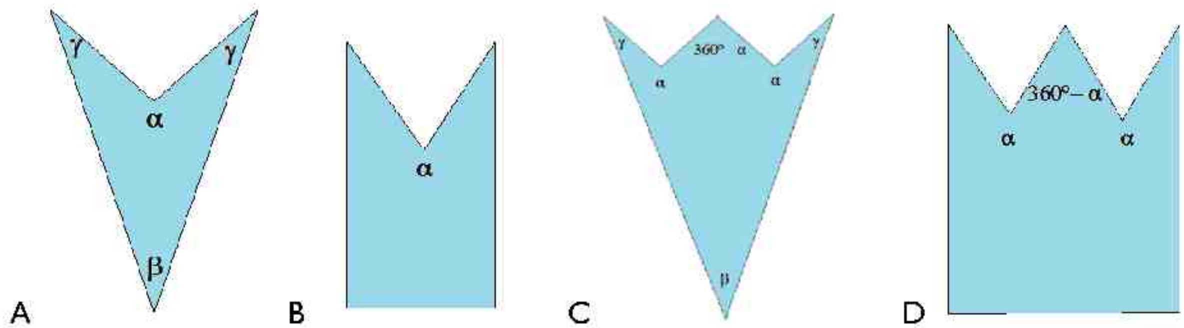


Figure 44. Symmetric tiles for n= 4, 5, 6, 7.

Notice that our symmetric tiles in Figure 44 above, have all caves identical and are on one side of the tile. We require each polygon to have every interior angle that forms a cave to measure $\alpha°$, and every interior angle between each cave to measure $(360 - \alpha)°$. Also, we require the edges forming each cave to be equal in length. In addition, in any symmetric n-gon where n is even, we require the interior angles $\gamma°$ and $\beta°$ to satisfy the condition $2\gamma° + \beta° = 360° - \alpha°$ Notice that the two symmetrical concave polygons in Figures 44A and 44C have a very similar V-like shape. If we take the symmetrical concave quadrilateral in Figure 44A and add two edges to form two caves within the original cave, we have a symmetrical concave polygon with $n = 6$ edges and two caves seen in Figure 44C. We could iterate this procedure for tiles A and B to obtain tiles with n-edges, for $n > 3$ even. We can use these polygons to tile the plane by arranging them in the following way as in Figure 45.

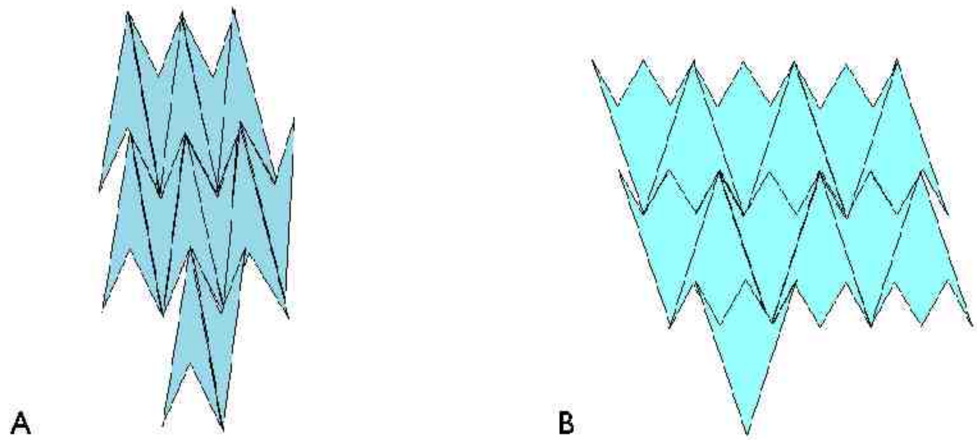


Figure 45. Tilings by symmetrical concave polygons with A) 4 edges and B) six edges that have only simple caves.

If we continue this process of adding two edges to the polygon in Figure 44C to form another cave, we again obtain a V-like polygon with $n = 8$ edges and three caves. We can arrange this V-like tile in the similar way as in Figure 45 to obtain the tiling below (Figure 46).

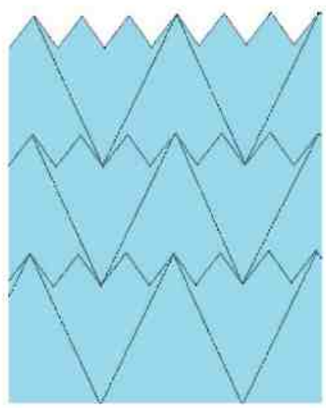


Figure 46. Tiling by symmetrical concave polygons with eight edges and three caves.

If we continue to add a set of two edges for caves as demonstrated, we can always form a V-like symmetrical concave polygon with an even number of edges that will tile the plane.

Now, consider the tiling seen in Figure 47 below by concave polygons that has 5 edges and one cave.

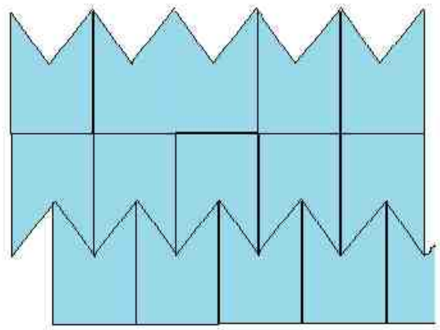


Figure 47. Tiling by symmetrical concave polygons with five edges and one cave.

If we take this five-sided symmetrical concave polygon and add two edges to form another cave within the original cave as before, we have formed a concave tile with seven edges and two caves. If we arrange these tiles in the similar way as we did for Figure 47, we generate the tiling seen below in Figure 48.

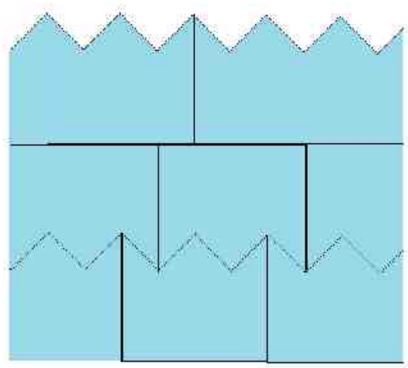


Figure 48. Tiling by symmetrical concave polygons with seven edges and two caves.

Notice that the method we are describing simply adds a set of two edges to form caves and does nothing to the base of the tile. This yield a V-like concave polygon if we have an even number of edges and a Castle-like concave polygon if our polygon has an odd number of edges. By repeating this method, we can generate any monohedral tiling with symmetrical concave polygon that has $n > 3$ edges.

Q.E.D.

Theorem 11: There exists a non-monohedral tiling of the plane by symmetrical concave polygons which have $n > 3$ edges and no bay cavity.

Proof: Consider a monohedral tiling with symmetrical concave polygon with even number of edges that have no bays from Theorem 10. We know for every polygon in the tiling, the interior angles that forms a cave must measure α° and the interior angles between each cave must be $(360 - \alpha)^{\circ}$. Also, the length of the edges forming the caves must be equal. Since this is the case, if we look at this monohedral tiling and remove a particular row of tile, we can replace this row with a different type of symmetrical concave polygons (scaling maybe required to make edges of the caves equal). We can choose to replace it with any V-like or Castle-like polygon with $n > 3$ edges described in Theorem 10. Consider the example in Figure 49 of a non-monohedral tiling given by symmetrical concave polygons with six edges and two caves along with symmetrical concave polygons that have eight edges and two caves.

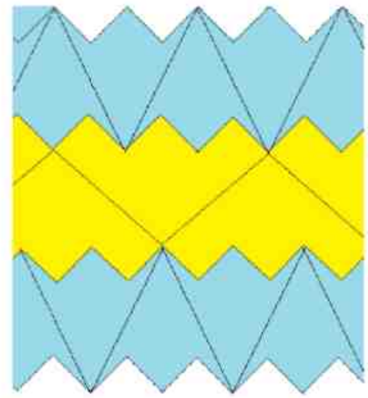


Figure 49.

We can also have use symmetrical polygon with an odd number odd edges as a replacement. Since a polygon with an odd number of edges has a flat base, we will need two rows of this polygon to replace the missing row. In Figure 50 on the left, we have an example of a non-monohedral tiling by symmetrical concave polygons with six edges and two caves along with symmetrical concave polygons with seven edges and two caves.

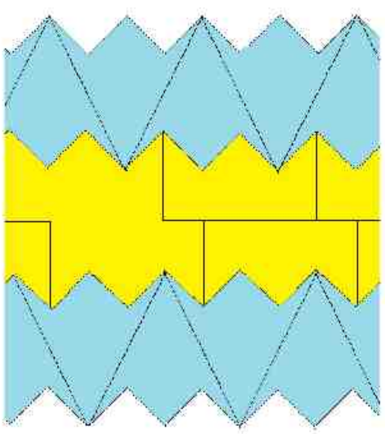


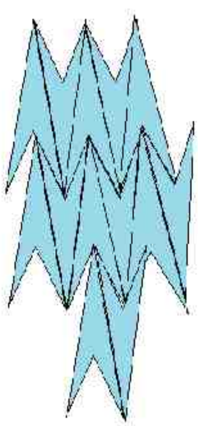
Figure 50.

As a generalization, if we take any monohedral tiling by our symmetrical concave polygons with even number of edges that has no bay and remove a row of tiles, we can replace this row with any symmetrical concave polygons with any number of edges greater than three and any number of caves.

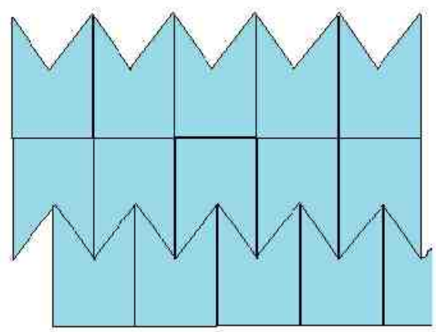
Q.E.D.

Theorem 12: It is not possible to tile the plane with any symmetrical concave polygons with $n > 8$ edges that have only one cave.

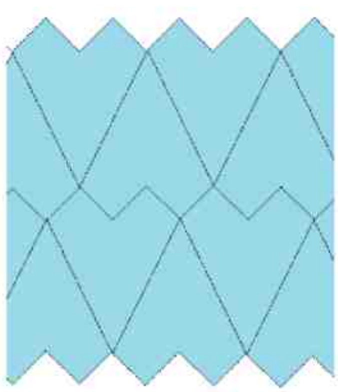
Proof: We will first show that for integer $n = 4, 5, 6, 7, 8$ there exist tilings by symmetrical concave n-gons with one simple cave (Figure 51).



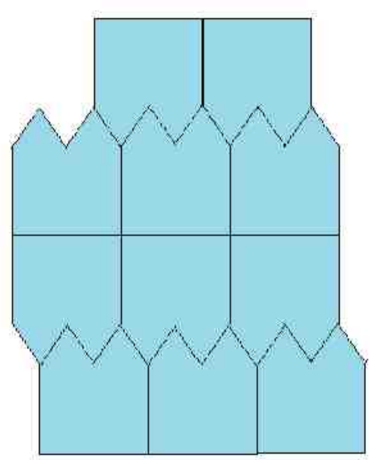
A) $n = 4$



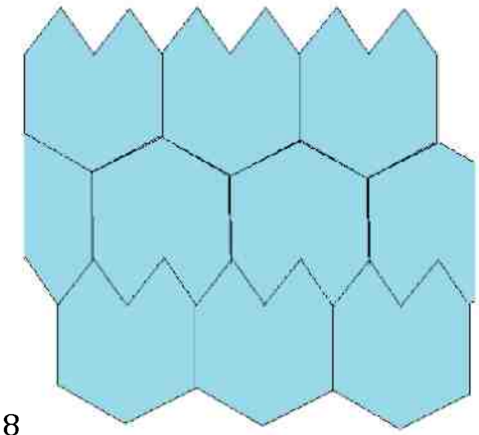
B) $n = 5$



C) $n = 6$



D) $n = 7$



E) $n = 8$

Figure 51. Tilings by symmetrical concave n-gons with one simple cave, for n= 4, 5, 6, 7, 8.

Now consider a symmetrical concave polygon with one simple cave that has nine edges and another with ten edges (Figure 52). Since these polygons are symmetrical, the symmetry line must pass through a cave and an edge for a polygon with an odd number of edges, and through a cave and a corner for a polygon with an even number of edges. Notice that the symmetry line divides the nine-sided as well as ten-sided concave polygons into two identical convex polygons with six edges (Figure 52). Hence, in order for the nine- and ten-sided symmetrical concave polygons to tile the plane, the convex six sided sub-polygons of these polygons must tile as well. In order for a six-sided convex polygon to tile, it must be one of the following Heesch's tile classification, *TTTTTT*, *TCCTCC*, $TG_1G_1TG_2G_2$, $TG_1G_2TG_1G_2$ *TCCTGG*, $C_3C_3C_3C_3C_3C_3$, $G_1CG_2G_1G_2C$ (Figure 39). These sequences of letters describe a tile's edges as we travel around its boundary. Each edge is assigned a letter describing its relation.

We will show that the convex sub-polygons for the ten and nine sided symmetrical concave polygons have no Heesch symmetries.

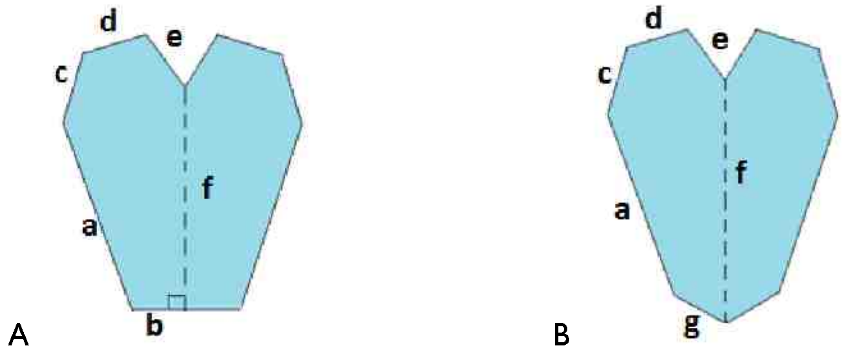


Figure 52. Symmetrical concave polygons with one cave that have A) nine and B) ten edges.

First we will analyze the convex sub-polygon in Figure 52A: We can eliminate TTTTTT since not all the edges can be translated. Notice that the next four Heesch symmetry types *TCCTCC*, $TG_1G_1TG_2G_2$, $TG_1G_2TG_1G_2$, *TCCTGG* requires that two opposite edges of the convex sub-polygon be related to each other by a translation. If we analyze the convex sub-polygon in Figure 52A, we can see that it is not possible to translate edge **b** onto edge **d** or similarly edge **c** onto edge **f**. Hence, the only pair of opposite edges that we have to analyze are edges **a** and **e**. From Figure 52A, we can see that it is possible to have a convex sub-polygon where edges **a** and **e** are related each other by a translation. However, if a convex sub-polygon has this particular Heesch symmetry type where edge **a** is a translation of edge **e**, any tiling by this sub-polygon would require that we place the second tile in the cavity of the original polygon such that edge **a** is edge-to-edge with edge **e**. We can see that this convex sub-polygon would not fit in the cavity of the original polygon due to the orientation of edge **b**. Therefore, we can eliminate all the Heesch types which have translations. Consider now the Heesch symmetry type $C_3C_3C_3C_3C_3C_3$ which requires that all the edges be related to an adjacent edge by a $120°$ rotation. Notice that our convex sub-polygon in Figure 52A must have a right angle. Therefore, edges **b** and **f** can never be related to each other by a $120°$ rotation. We can see it is possible however, to have a convex sub-polygon where the pair of edges **a** and **b**, **c** and **d**,

e and **f** are related by a $120°$ rotation. A tiling by this convex sub-polygon would require that we place the second tile in an orientation such that edges **e** and **f** are edge-to-edge. Looking at the original polygon in Figure 52A, we can see that this is not possible since edge **f** is edge-to-edge with itself. Hence we can eliminate symmetry type $C_3C_3C_3C_3C_3C_3$. Lastly, the symmetry type $G_1CG_2G_1G_2C$ requires that our convex sub-polygon have a pair of opposite edges which can be glided onto each other. Since we are gliding straight edges, this is equivalent to translating them. We have already shown prior that this is not possible. Therefore, we cannot tile the plane with polygon in Figure 52A.

Now we will analyze the convex sub-polygon in Figure 52B : We can eliminate symmetry type TTTTTT as not all the edges can be translated. Again, the next four Heesch symmetry types $TCCTCC$, $TG_1G_1TG_2G_2$, $TG_1G_2TG_1G_2$, $TCCTGG$ requires that two opposite edges of the convex sub-polygon be a translation of each other. If we analyze the convex sub-polygon in Figure 52B, we can see that edge **g** cannot be translated onto edge **d** and edge **c** cannot be translated to edge **f**. However, it is possible to have a convex sub-polygon such that edges **a** and **e** are related by a translation. If the convex sub-polygon in Figure 52B has this particular Heesch symmetry type where edges **a** and **e** are related by a translation, by the same reasoning stated prior, any tiling by this convex sub-polygon would require that we place the second tile in the cavity of the original polygon, such that edge **a** is edge-to-edge with edge **e**. We can see that this again is impossible since the convex sub-polygon would not fit in cavity of the original polygon. Hence, we can eliminate all Heesch types which have translations. Consider now the Heesch symmetry type $C_3C_3C_3C_3C_3C_3$. We can see that the sub-polygon in Figure 52B is similar to the sub-polygon in Figure 52A. By the same argument made earlier, we can eliminate symmetry type $C_3C_3C_3C_3C_3C_3$. The last Heesch type $G_1CG_2G_1G_2C$ requires that our convex sub-polygon have a pair of opposite edges which can be glided onto each other. A glide between two edges is equivalent to a translation since our polygon have only straight edges. We have already shown that any Heesch type with translation is not possible. Therefore, we cannot tile the plane with polygon in Figure 52B. We can generalize and see that if a symmetrical concave polygon with more than ten edges has one cave, then it will have two identical convex sub-polygons with $n > 6$ edges. By Theorem 7, we can say that the convex sub-polygon cannot tile the plane. Therefore, the symmetrical concave polygon itself will not tile the plane.

Q.E.D.

Theorem 13: For each integers $n > 6$, there exists a monohedral tiling of the plane by a concave n-gon with a bay.

Proof: We will show that there does not exist a monohedral tiling by symmetrical concave 5-gon and 6-gon with a bay. First consider the labeled symmetrical concave 5-gon with the maximum number of bays below (see Figure 53A). According to Heesch's table in Figure 39, this tile must have one of the following Heesch tile classification to tile the plane, *TCTCC*, $CC_4C_4C_4C_4$, *TCTGG*, $CC_3C_3C_3C_3$, $CG_1G_2G_1G_2$. We will show that this polygon does not have any of Heesch tile classifications.

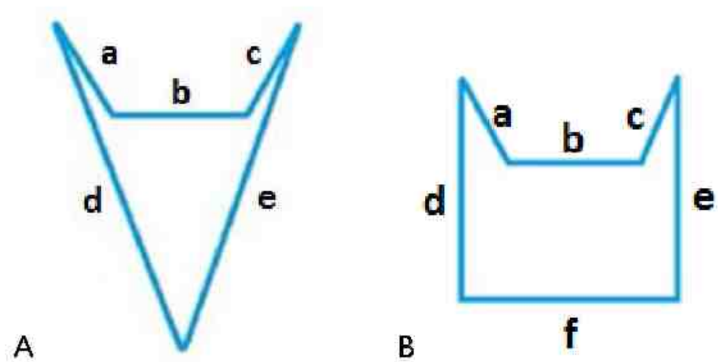


Figure 53. A) Concave 5-gon with one bay and B) 6-gon with one bay.

Looking at the polygon in Figure 53A, we can see that it is not possible to have any translations between two opposite edges. Hence, we can eliminate the Heesch types *TCTCC, TCTGG*. We can also eliminate Heesch symmetry type $CG_1G_2G_1G_2$ which requires that a pair of opposite edges be glided (translated since we have straight edges) onto each other, which is not possible. From Figure 53A, we can also see that only adjacent edges **d** and **e** can be related to each other by a 90° rotation. If this is this is true, then according to Heesch type $CC_4C_4C_4C_4$, either edges **a** and **b** or **c** and **b** must be related by a 90° rotation. Analyzing the polygon, we can see that this is impossible since edges **a**, **b**, and **c** forms the bay and can never be related to one another by a 90° rotation. Hence we can eliminate Heesch type $CC_4C_4C_4C_4$. Lastly, Heesch symmetry type $CC_3C_3C_6C_6$ requires that two adjacent pair of edges be related by a 120° rotation, followed by another pair of edges related by a 60° rotation using the interior angles. Consider the pair of adjacent edges **d** and **e,** which we can see from Figure 53A is the only pair of edges that can be related to each other by a 120° or 60° rotation. If edges **d** and **e** are related by a 120° rotation, then either edges **a** and **b** or edges **c** and **b** must be related by a 60° rotation. Since the interior angles formed by edges **a** and **b** as well as **c** and **b** are greater than 180°, this condition is not possible to have. Similary, if edges **d** and **e** are related by a 60° rotation, then either edges **a** and **b** or edges **c** and **b** must be related by a 120° rotation, which again is impossible. Hence, we can eliminate Heesch type $CC_3C_3C_6C_6$ and conclude that the polygon if Figure 53A will not tile the plane.

Now consider the 6-gon with the maximum number of bays in Figure 53B. To tile the plane, this polygon must have at least one of the following Heesch symmetry types *TTTTTT, TCCTCC*, $TG_1G_1TG_2G_2$, $TG_1G_2TG_1G_2$, *TCCTGG*, $C_3C_3C_3C_3C_3C_3$, $G_1CG_2G_1G_2C$. We will show that this polygon have no Heesch symmetries listed. First we can eliminate symmetry type TTTTTT as not all the edges can be translated. Consider the next four Heesch symmetry types *TCCTCC*, $TG_1G_1TG_2G_2$, $TG_1G_2TG_1G_2$, *TCCTGG* which requires that the polygon have a pair of opposite edges related to each other by a translation. From Figure 53B, we can see that it is impossible to relate opposite edges **d** and **c**, **b** and **f**, **a** and **e** by a translation. Hence, we can eliminate all Heesch symmetry types which have translations. Notice that in Figure 53B, edge **d** can never related to edge **a** by a $120°$ rotation and similarly edge **c** can never be related to edge **e** by a $120°$ rotation. Since not all the edges can be rotated $120°$ onto an adjacent edge, we can eliminate the symmetry type $C_3C_3C_3C_3C_3C_3$. Lastly, we can eliminate Heesch type $G_1CG_2G_1G_2C$ since any pair of opposite straight edges related by a glide is equivalent them being related by a translation, which we have already shown is not possible for this polygon. Hence, the polygon in Figure 53B cannot be use to tile the plane.

We will now show that it is possible to construct a monohedral tiling of the plane by concave n-gon with a bay, for $n > 6$. Consider the construction method used previously in the proof of Theorem 9. We will modify this method slightly so that we can generate a monohedral tiling with a bay. First, we start with base tiling given by triangles, and like before, we will substitute each edge around a given vertex with two edges to generate a tiling with concave 5-gon (see Figure 40). Now, we will repeat the method by substituting one of the edges of a cave with two edges. However, we will substitute in these two edges in the same orientation as we did in the construction of the tiling in Figure 40. Doing so, we will generate a tiling with concave 7-gon with one bay (see Figure 54).

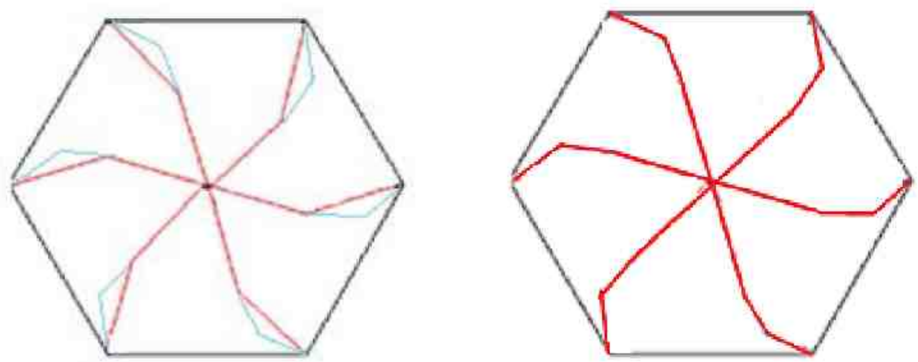


Figure 54. Construction of a monoherdral tiling by concave 7-gon with one bay.

By iterating this construction method, we will find all the monohedral tilings by concave n-gon with a bay, when $n > 6$ is odd. Similarly, we can apply this construction method to the base tiling by squares to generate all the monohedral tiling by concave n-gon with a bay for even $n > 6$.

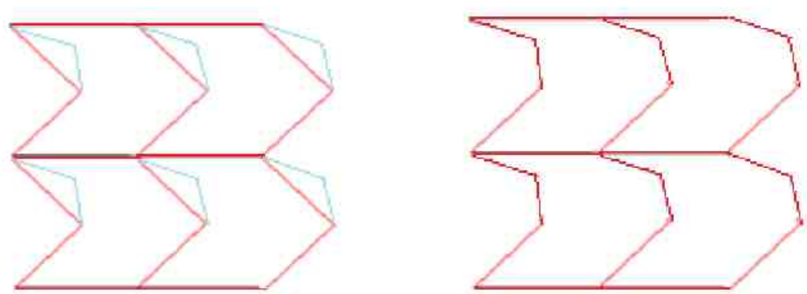


Figure 55. Construction of a monohedral tiling by concave 8-gon with a bay.

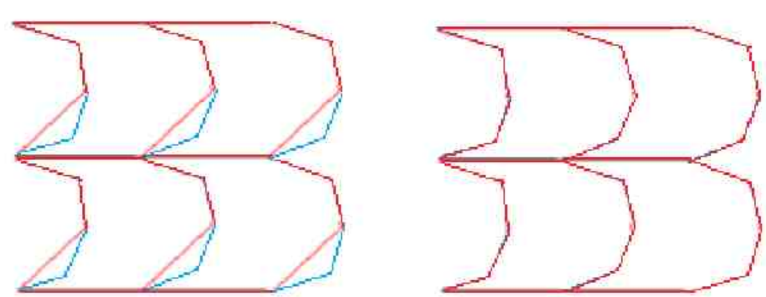


Figure 56. Construction of a monohedral tiling by concave 10-gon with a bay.

Q.E.D.

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