The Docking Problem:
Optimal Rendezvous and Docking Trajectories for Lunar Exploration

By

Alba L. Romero

A DISSERTATION Submitted in partial fulfillment of the requirements for the degree of

Masters in Science
(Mathematics)

at the
CALIFORNIA STATE UNIVERSITY - CHANNEL ISLANDS
2014
Signature page for the Masters in Mathematics Thesis of
Alba L. Romero

APPROVED FOR THE MATHEMATICS PROGRAM

Dr. Ivona Grzegorczyk, Thesis Advisor  Date 5/21/14

Dr. Gregory Wood, Thesis Committee  Date 5/21/14

APPROVED FOR THE UNIVERSITY

Dr. Gary A. Berg, AVP Extended University  Date 5/21/14
Non-Exclusive Distribution License

In order for California State University Channel Islands (CSUCI) to reproduce, translate and distribute your submission worldwide through the CSUCI Institutional Repository, your agreement to the following terms is necessary. The author(s) retain any copyright currently on the item as well as the ability to submit the item to publishers or other repositories.

By signing and submitting this license, you (the author(s) or copyright owner) grants to CSUCI the nonexclusive right to reproduce, translate (as defined below), and/or distribute your submission (including the abstract) worldwide in print and electronic format and in any medium, including but not limited to audio or video.

You agree that CSUCI may, without changing the content, translate the submission to any medium or format for the purpose of preservation.

You also agree that CSUCI may keep more than one copy of this submission for purposes of security, backup and preservation.

You represent that the submission is your original work, and that you have the right to grant the rights contained in this license. You also represent that your submission does not, to the best of your knowledge, infringe upon anyone's copyright. You also represent and warrant that the submission contains no libelous or other unlawful matter and makes no improper invasion of the privacy of any other person.

If the submission contains material for which you do not hold copyright, you represent that you have obtained the unrestricted permission of the copyright owner to grant CSUCI the rights required by this license, and that such third party owned material is clearly identified and acknowledged within the text or content of the submission. You take full responsibility to obtain permission to use any material that is not your own. This permission must be granted to you before you sign this form.

IF THE SUBMISSION IS BASED UPON WORK THAT HAS BEEN SPONSORED OR SUPPORTED BY AN AGENCY OR ORGANIZATION OTHER THAN CSUCI, YOU REPRESENT THAT YOU HAVE FULFILLED ANY RIGHT OF REVIEW OR OTHER OBLIGATIONS REQUIRED BY SUCH CONTRACT OR AGREEMENT.

The CSUCI Institutional Repository will clearly identify your name(s) as the author(s) or owner(s) of the submission, and will not make any alteration, other than as allowed by this license, to your submission.

Title of Item: The Ducking Problem: Optional Rendezvous and Docking Trajectories for Lunar Exploration

3 to 5 keywords or phrases to describe the item: Rendezvous, docking, space travel, trajectories, orbits

Authors Name (Print): Alba Romero

Author's Signature: [Signature]

Date: 5/7/14

This is a permitted, modified version of the Non-exclusive Distribution License from MIT Libraries and the University of Kansas.
ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor Dr. Ivona Grzegorczyk from California State University, Channel Islands for her excellent guidance, patience, advice, and encouragement. It was truly an amazing experience working with her and I could not have thought of a more fitting thesis topic. Without her guidance and constant help this thesis would not have been possible.

I would also like to thank Dr. Gregory Wood for being a part of the committee and for reading and commenting on the manuscripts. In addition, a thank you to Dr. Cindy Wyels for introducing me to Mathematics research during the 2009 Summer REU and for encouraging me to pursue a graduate degree.

A special thanks to my family. Words cannot express how grateful I am to my mother, father, and brothers for all of the sacrifices that you have made on my behalf. I am truly fortunate to have such loving family and friends. I appreciate their love, advice, and continued support in my educational endeavors. Para mis amores, los quiero mucho!

NASA, here I come.

Camarillo, California

May 15, 2014
The moon landing in 1969, achieved despite of the complexity and the lack of computer power at the time, raises questions about various computational aspects of the mission. We present a model of the lunar docking problem, in which the optimal rendezvous orbits to dock the lunar module back onto the orbiting command module are presented through analytical equations and analysis for the optimal trajectories.
# TABLE OF CONTENTS

1 **Introduction** 4

1.1 History ........................................... 4
1.2 Mission Overview ................................... 7
1.3 The Spacecraft .................................... 9

2 **Preliminary Definitions** 14

2.1 Introductory physics set up ......................... 14

3 **Ellipse in Polar Coordinates** 21

3.1 The Ellipse ....................................... 21

4 **Orbits & Calculations** 33

4.1 Model for Potential Rendezvous ................... 33
4.2 Potential Rendezvous ............................... 34
4.3 Elliptical Trajectories ............................. 34
4.4 The Copernican Model ............................. 40

5 **Our Model** 44
# TABLE OF CONTENTS

5.1 Hypothetical Mission ........................................ 44
5.2 Match of Velocities ........................................... 46
5.3 Launch Angle .................................................. 49

6 Conclusions ...................................................... 60

Bibliography ........................................................ 63
1.1 History

The space race between the United States and the Soviet Union began in the late 1950’s. The rivalry between the two nations led to successful and unsuccessful satellite and rocket orbit missions. Initially, the Soviets led the space race. They tested their rockets by sending animals (mainly dogs and chimps) on the first orbit missions. In 1961, the Soviet Union accomplished the first human spaceflight with Vostok 1, which made a single orbit around the Earth. Other notable firsts of the Soviet space program (between 1957 and 1969) include: first satellite in orbit, first animal to orbit Earth, first woman in space, and the first probe in lunar orbit.
1.1. HISTORY

The United States was not too far behind. Some notable firsts for the U.S. include: first communications satellite (1958), first piloted spacecraft orbit exchange (1965), first rendezvous in space (1965), first human crewed spaceflight and orbit of the Moon (1968). Finally on July 20, 1969, the United States completed a large feat with the first successful manned mission to the Moon with Apollo 11. Since the Moon landing was a large step forward, some believed that either many (or all) aspects of the Moon landing had been fabricated due to oddities, inconsistencies in images, and other environmental or technological features.

![Figure 1.1: Buzz Aldrin explores the lunar surface, Credit: NASA](image)

Figure 1.1: Buzz Aldrin explores the lunar surface, Credit: NASA
CHAPTER 1. INTRODUCTION

One prominent conspiracy theory stemmed from the lack of computing power at the time, simple computers could not model landing and returning trajectories accurately. 64k machines made calculating rendezvous between the command/service module (CSM) and the lunar module (LM) a daunting task (see Figure 1.2). Hence, it is believed that the entire Moon landing mission would have been close to impossible due its complexity. This paper presents a model of the docking problem, in which the optimal rendezvous orbits to dock the LM back onto the orbiting CSM are sought in analytical forms.

Figure 1.2: Apollo Guidance Computer, Credit: NASA
1.2 Mission Overview

1969’s Apollo 11’s mission objective was to "Perform manned lunar landing and return mission safely." Achieving said objective was no easy task; in this chapter we outline and describe the journey and spacecraft used in the Apollo 11 mission. First we will look at the three main flight plans that NASA considered to get to the Moon, outline the flight plan chosen for this particular mission, and finally we look at the spacecraft that made the Moon landing possible.

1.2.1 Flight Plans

When planning the trip to the Moon, NASA considered three possible flight plans: direct ascent, Earth orbital rendezvous (EOR), and lunar orbital rendezvous (LOR), Figure 1.3; detail the three contending flight plans [1].

The direct ascent approach is just that, it proposed that a powerful enough rocket be built to reach the Moon directly. This rocket would be launched from Earth and land directly on the Moon. Once the lunar expedition was complete, the crew would return in the same rocket. This approach was thrown out since the rocket would have to be very large and could not be completed
by the president's deadline. Hence the other two options were to be considered.

Figure 1.3: The three contending flight plans: direct ascent (left), EOR (center), LOR (right) Credit: NASA

The **EOR** approach proposed that a spacecraft, launched from Earth, be placed on an orbit around the Earth to be refueled. Once refueled, the spacecraft would continue on its journey to the Moon. One advantage of EOR was that no large rocket would have to be developed for the mission. Despite this advantage, the third flight plan was chosen.
1.3. THE SPACECRAFT.

The LOR approach was proposed by John C. Houbolt (of the Langley Research Center) in 1962. This approach would send a three-module spaceship into orbit around the Moon. One module would detach to make the landing onto the Moon and the other two modules (command/service modules) would remain attached in lunar orbit. Once done on the lunar surface, the crew would leave the Moon's surface in the lunar module and dock onto the spacecraft in lunar orbit. The lunar and service modules would later be discarded, while the command module would take the crew back to Earth. In the following sections, we will look at LOR and the spacecraft used by the U.S. for the Moon landing.

1.3 The Spacecraft

We start with some technical information on the flying equipment used at the time of the first Moon landing. The LOR flight plan called for specific equipment and machinery to be built [2]. The launch vehicle for the Apollo mission was the Saturn V rocket. This rocket was a three-stage (see Definition 2.1.4) launch vehicle designed to send the Apollo spacecraft into space. Each stage would burn its engines until it ran out of fuel and then be detached and dis-
carded. Then the engines of the next stage would fire, sending the rocket further along into space [3].

### 1.3.1 Saturn V

Since NASA selected the LOR flight plan, it needed a rocket powerful enough to send a three man crew in a three-module spaceship into orbit around the Moon: the Saturn V rocket. It was about 111 meters tall (from base to top). Fueled, the Saturn V weighed 2.8 million kilograms (6.2 million pounds). At launch, it generated 34.5 million newtons (7.6 million pounds) of thrust [3].

![Saturn V Launch Vehicle](attachment:Saturn_VLaunchVehicle.png)

**Figure 1.4:** Diagram of Saturn V Launch Vehicle, *Credit: NASA*
1.3. THE SPACECRAFT.

The Saturn V’s first stage carried 203,400 gallons of kerosene fuel and 318,000 gallons of liquid oxygen that was necessary for combustion at liftoff. At an altitude of 42 miles (67 kilometers) the first stage was jettisoned and the second stage, which carried 260,000 gallons of liquid hydrogen fuel and 80,000 gallons of liquid oxygen, went into effect. Next, the second stage was discarded and the third stage’s rocket engine was fired. The third stage carried 66,700 gallons of liquid hydrogen fuel and 19,359 gallons of liquid oxygen. The third stage’s engine was fired until the rocket reached a sufficient speed to reach Earth orbit [4]. The third stage also provided the final push into space; it was responsible for sending Apollo 11 to the Moon at 24,000 mph. Finally, the third stage was discarded once the CSM coupled with the LM were detached [5]. See also our calculations in 5.3.

1.3.2 Service Module

The service module (SM) was a nearly 25 feet high and 13 feet in diameter, cylindrical unit mounted directly behind the CM. It weighed approximately 55,000 pounds with propellant and 11,500 pounds when empty (i.e. 80% was fuel). The SM contained the principal propulsion system, propellant, electric system, water, as well as other necessary supplies.
CHAPTER 1. INTRODUCTION

1.3.3 Command Module

The command module (CM) was a conical unit nearly 11 feet high and 13 feet in diameter at its base. The CM was located directly above the SM. The crew would spend the majority of the journey in this unit. The CM served as the flight control center, living quarters, and re-entry vehicle. Detailed Command Module specifications in metric and English systems (note that it was quite small), see Figure 1.5 [6]:

- Height: 3.2 m (10 ft 7 in)

- Maximum Diameter: 3.9 m (12 ft 10 in)

- Weight: 5,900 kg (13,000 lb)

Figure 1.5: Command/Service Module and Lunar Module, Credit: NASA
1.3. **THE SPACECRAFT.**

### 1.3.4 Lunar Module

During the flight, the lunar module (LM) was housed in a 28 feet tapered cylinder - the lunar module adapter (SLA). The SLA was between the SM and the instrument unit above the third stage of the Saturn V carrier. The LM was to be used to send two astronauts onto the lunar surface. The LM would also serve as living quarters, communications center, and base of exploration. After 21 hours and 36 minutes on the lunar surface, the LM lifted off the Moon. Lunar Module Specifications [6], [7]:

- **Weight (empty):** 3920 kg (8650 lb)
- **Weight (with Crew & Propellant):** 14,700 kg (32,500 lb)
- **Height:** 7.0 m (22 ft 11 in)
- **Width:** 9.4 m (31 ft 00 in),
- **Descent Engine Thrust:** 44,316 Newtons (9870 lb) maximum, 4710 Newtons (1050 lb) minimum
- **Ascent Engine Thrust:** 15,700 Newtons (3500 lb)
- **Fuel:** 50-50 mix of Unsymmetrical Dimethyl Hydrazine (UDMH) & Hydrazine
Chapter 2

Preliminary Definitions

2.1 Introductory physics set up

Newton's law of gravity: The vector attractive force \((F_{12})\), one body \((m_1)\) exerts on another \((m_2)\) is given by:

\[
F_{12} = -G \frac{m_1 m_2}{|r_{12}|^2} \hat{r}_{12}
\]

where \(m_1\) and \(m_2\) are the masses of the respective bodies, \(|r_{12}| = |r_2 - r_1|\) is the distance between the two bodies, and \(\hat{r}\) is the unit vector pointing to \(m_2\) from \(m_1\).

Definition 2.1.1. The universal gravitational constant is the constant in Newton’s law of gravitation that is used in fundamental equations for spacial cal-
calculations. (Note that $G$ does not depend on any interacting bodies). Recent measurements show the value of $G$ as: $G = 6.67384(80) \times 10^{-11} m^3 kg^{-1} s^{-2}$.

Note in 1969, the approximation of the gravitational constant was $G = 6.6732(31) \times 10^{-11} m^3 kg^{-1} s^{-2}$ [8].

![Figure 2.1: Illustration of escape velocity. Projectiles A and B fall back to Earth. Projectile C achieves a circular orbit. Projectile D achieves an elliptical orbit. Projectile E escapes.](image)

**Definition 2.1.2.** The escape velocity is the speed needed for an object to break free from the gravitational attraction of a massive body, without further propulsion and is given by:

\[
V_e = \sqrt{\frac{2Gm_1}{r}}
\]

(2.1)

In Equation 2.1, $G$ is the universal gravitational constant, $m_1$ is the mass of the planet or celestial body, and $r$ is the distance from the center of gravity.
As seen in Figure 2.1, an object will fall back to the massive body if the velocity is too low, circulate around the mass for some range of velocities and above certain velocities it will escape.

**Definition 2.1.3.** *Thrust* is the forward or upward force produced by the engines of a plane or spacecraft as a result of burned fuel.

**Definition 2.1.4.** A *stage* is a section of a rocket containing a rocket engine or a cluster of rocket engines, where the stages are typically separated and discarded when its fuel is used.

### 2.1.1 Escape velocities for Moon and Earth

The escape velocities for the Moon and Earth are important to our docking model. We start by calculating the Earth’s escape velocity. Necessary in these calculations are the mass of the planet, its radius, and $G$.

Earth’s Mass: $5.97 \times 10^{24}$ kg  
Earth’s Radius: $6.38 \times 10^6$ m

$G = 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}$

Hence the escape velocity of the Earth is:

$$v_e = \sqrt{\frac{2Gm_1}{r}} = \sqrt{\frac{2(6.67)(10^{-11})(5.97)(10^{24})}{6.38(10^{6})}} = 11.2 \text{ km/s}$$
2.1. INTRODUCTORY PHYSICS SET UP

Now we calculate the Moon’s escape velocity.

Moon’s Mass: $7.3477 \times 10^{22} \text{ kg}$ \hspace{1cm} Moon’s Radius: $1.7371 \times 10^6 \text{ m}$

$G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$

Hence the escape velocity of the Moon is:

$$v_e = \sqrt{\frac{2Gm_1}{r}} = \sqrt{\frac{2(6.67)(10^{-11})(7.3477)(10^{24})}{1.737(10^6)}} = 2.38 \text{ km/s}$$

Note that the escape velocity of the Moon is almost 5 times smaller than the escape velocity of the Earth.

2.1.2 Kepler’s laws of planetary motion

Prior to Kepler, in 1543 Nicolas Copernicus proposed a heliocentric solar system model in which the Sun was placed at the center of the solar system and the planets revolved in circular orbits around the Sun [9]. More details on the Copernican model will be discussed in Section 4.4. At the beginning of the 17th century, Kepler refined these ideas to the more accurate solar system model. His work resulted in the formulation of Kepler’s laws of planetary motion. These three laws of planetary motion improved the Copernican model and describe the movement of a body around a large mass [10].
CHAPTER 2. PRELIMINARY DEFINITIONS

Figure 2: Earth's elliptical trajectory.

- **Kepler's first law**: The orbit of every body traveling around a given mass is an ellipse with the mass as one of its foci.

Figure 2.2 shows an example of the Earth orbiting the Sun. Note that the two foci lie on one of the axis of the ellipse. Therefore, we expect spacecraft to follow elliptical orbits while no additional forces (like thrust, see 2.1.3) are present.

- **Kepler's second law**: A line joining a planet and the Sun sweeps out equal areas during equal intervals of time, see Figure 2.3.

Since the orbiting object sweeps areas $A$ and $B$ in equal periods of time, the speed of the object changes depending on its distance from the center of the
Sun. Hence, linear speed is greatest at the point closest to the Sun and slowest at the point farthest from the Sun. This arises from the conservation of angular momentum, see Page 25.

![Diagram of elliptical orbit with A and B as points and Sun at one focus]

Figure 2.3: Equal areas, A and B in equal times.

- **Kepler’s third law**: The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit, see Figure 2.2.

\[ p^2 \propto a^3 \]

Using the third law, we are able to calculate the time for the full orbit around a large mass.

**Definition 2.1.5.** *Perilune* is the point at which an object (a rocket) orbiting the Moon, is nearest to the Moon, see Figure 2.4.
Definition 2.1.6. **Apolune** is the point at which an object orbiting the Moon is the farthest to the Moon's center, see Figure 2.4.

**Figure 2.4**: Apolune and Perilune
3.1 The Ellipse

In this chapter, we study elliptical orbits and describe different mathematical models commonly used. Consider an ellipse with the following standard equation in \( x, y \) coordinate system:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]  

Equation 3.1 defines an ellipse symmetric in \( x \) and \( y \) axis. Sometimes it is more convenient to use polar coordinates to describe ellipses. In orbital mechanics, we measure the radius, \( r \), and the angle, \( \theta \), from a focus \( F \) (located at the origin) instead of the center (Figure 3.1). An ellipse in polar coordinates is
CHAPTER 3. ELLIPSE IN POLAR COORDINATES

defined by:

\[ x = c + r \cos \theta \quad \text{and} \quad y = r \sin \theta \]  \hspace{1cm} (3.2)

Where \( c \) shifts one of the foci of the ellipse to \((0,0)\).

Figure 3.1: Ellipse with \( F \) as primary focus.

By substituting Equation 3.2 into Equation 3.1, we get:

\[
\frac{(c + r \cos \theta)^2}{a^2} + \frac{r \sin \theta)^2}{b^2} = 1
\]

\[
b^2(c + r \cos \theta)^2 + a^2(r \sin \theta)^2 = a^2b^2
\]

\[
b^2c^2 + 2bc \cdot r \cos \theta + b^2r^2 \cos^2 \theta + a^2 \cdot r^2 \sin^2 \theta = a^2b^2
\]

Substituting \( \sin^2 \theta = 1 - \cos^2 \theta \), we get:

\[
b^2c^2 + 2b^2cr \cos \theta + b^2r^2 \cos^2 \theta + a^2r^2 - a^2r^2 \cos^2 \theta = a^2b^2 \]  \hspace{1cm} (3.3)

We will use the more convenient notation here and define the eccentricity constant as \( e = c/a \), which gives \( c = ae \). We know that \( b^2 = a^2 - c^2 \), hence by
substituting and simplifying we get: \( b^2 = a^2 - a^2e^2 = a^2(1 - e^2) \). Now, substituting for \( b^2 \) and \( c \) into Equation 3.3, we get:

\[
a^2(1-e^2)a^2e^2 + 2ae(a^2(1-e^2))r\cos\theta + a^2(1-e^2)r^2\cos^2\theta + a^2 - a^2 r^2 \cos^2 \theta = a^2(a^2-1 - e^2)
\]

And finally, simplifying:

\[
(1-e^2)a^2e^2 + 2ae(1-e^2)r\cos\theta + (1-e^2)r^2\cos^2\theta + r^2 - r^2 \cos^2 \theta = a^2(1-e^2)
\]

\[
r^2 = |er\cos\theta - a(1-e^2)]^2
\]

\[
r = \pm[er\cos\theta - a(1-e^2)]
\]

Therefore, the ellipse in polar coordinates can be described as (we take the positive root, \( r > 0 \) here):

\[
r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{3.4}
\]

This equation is very useful for calculating the position of the orbiting body around the mass centered at \((0,0)\) and based on the given data. Now we state useful equations for circular and elliptical orbits that are very helpful for orbital calculations [11], [12]:

- The length from apolune to perilune is twice the major axis: \( r_a + r_p = 2a \)

- The length from the primary focus to perilune: \( r_p = r_{min} = a(1-e) \)
CHAPTER 3. ELLIPSE IN POLAR COORDINATES

- The length from the primary focus to the apolune: \( r_a = r_{max} = a(1 + e) \)

- From the gravitational two-body problem, \( \mu = G(m_1 + m_2) \)

- With \( m_1 \gg m_2, \mu \equiv Gm_1 \) (gravitational parameter of the larger body).

- The eccentricity: \( e = \frac{r_a - r_p}{r_a + r_p} = \sqrt{1 + \frac{2EL^2}{\mu^2}} \)

- The magnitude of angular momentum: \( L = rV \cos \phi \)

- The velocity along a circular orbit: \( V = \sqrt{\frac{\mu}{r}} \)

- The velocity of an object following a specified trajectory: \( V = \sqrt{2(E + \frac{\mu}{r})} \)

The total specific mechanical energy, \( E \), at a given time (energy per unit mass) is the sum of the specific kinetic energy \( \frac{V^2}{2} \) and the specific potential energy \( \frac{\mu}{r} \).

- Total specific energy: \( E = \frac{V^2}{2} - \frac{\mu}{r} \).

- Circular orbit energy: \( E = -\frac{\mu}{2a} \).

Hence, by substituting \( p = a(1 - e^2) = \frac{L^2}{\mu} \) in Equation 3.4 we obtain another equation describing elliptical orbits:

\[
r = \frac{p}{1 + e \cos \theta}
\]
At apolune and perilune, the elevation angle $\phi = 0$. At this point the magnitude of the angular momentum, $L$, is the product of the distance from the primary focus and the respective velocity at each point. Hence we can calculate the velocities at perilune and apolune as follows:

$$L = r_p V_p = r_a V_a$$

$$V_p = \frac{L}{r_p} = \frac{r_a V_a}{r_p} = V_{max}$$

$$V_a = \frac{L}{r_a} = \frac{r_p V_p}{r_a} = V_{min}$$

Here we show two basic examples on how the previous equations are applied and utilized in orbital mechanics.

**Example 3.1.1.**
Consider an Earth-surface circular satellite. We find:

(a) orbital velocity,

(b) the energy,

(c) the angular momentum, and

(d) the period.

**Solution.** Since the orbit is circular, there is no apogee or perigee, the primary focus is located at the center of the Earth, the semimajor axis is equal to the radius, the orbital velocity, $V$, is constant, and the elevation angle, $\phi$, is
always zero. Also known: eccentricity \( e = 0 \), height of the satellite \( h = 0 \),

\[ r = r_E = 6.378 \times 10^6 \text{ m} = 6378 \text{ km}, \mu_E = 3.986 \times 10^{14} \text{ m}^3/\text{s}^2, \, g_0 = 9.80 \text{ m/s}^2 \]

(standard value at sea level).

(a) Orbital Velocity \( V = \sqrt{\frac{\mu}{r}} \).

\[ V = \sqrt{\frac{3.986 \times 10^{14}}{6.378 \times 10^6}} = 7905 \text{ m/s} = 28,460 \text{ km/h} \]

(b) Energy \( E = -\frac{\mu}{2a} \).

\[ E = -\frac{3.986 \times 10^{14}}{2 \times (6.378 \times 10^6)} = -3.125 \times 10^7 \text{ m}^2/\text{s}^2 \]

(c) Angular momentum \( L = rV \cos\phi \) (with \( \phi = 0^\circ \)).

\[ L = (6.378 \times 10^6) \times 7905 = 5.042 \times 10^{10} \text{ m}^2/\text{s} \]

(d) Period \( P = 2\pi \sqrt{\frac{r^3}{\mu}} \), the time to complete one orbit.

\[ P = 2\pi \sqrt{\frac{(6.378 \times 10^6)^3}{3.986 \times 10^{14}}} = 5069 \text{ s} = 84.4 \text{ min} \]

\[ \square \]

It turns out that on the Earth-centered (geocentric) orbits, the Earth-surface circular satellite has the lowest energy, shortest period, and the highest velocity. Hence, it is referred to as the minimum-energy minimum-period geocentric orbit [11]. For these reasons, circular orbits are very interesting for space
3.1. THE ELLIPSE

travel.

Next, let us observe what happens when we increase the altitude of the satellite to 600 km and redo Example 3.1.1.

**Example 3.1.2.**

Consider satellite orbiting Earth at an altitude of 600 km. We find:

(a) orbital velocity,

(b) the energy,

(c) the angular momentum, and

(d) the period.

**Solution.** Known: height of the satellite = $h = 600 \text{ km}$, $r = r_E + 600 \times 10^3 = 6.978 \times 10^6 \text{ m}$.

(a) Orbital Velocity = $V = \sqrt{\frac{\mu}{r}}$.

$$V_{cs} = \sqrt{\frac{3.986 \times 10^{14}}{6.978 \times 10^6}} = 7557.96 \text{ m/s} = 27,208.7 \text{ km/h}$$

(b) Energy = $E = -\frac{\mu}{2a}$.

$$E = -\frac{3.986 \times 10^{14}}{2 \times (6.978 \times 10^6)} = -2.856 \times 10^7 \text{ m}^2/\text{s}^2$$
(c) Angular momentum \( L = r V \cos \phi \) (with \( \phi = 0^\circ \)).

\[
L = (6.978 \times 10^6) \times 7557.96 = 5.27 \times 10^{10} \text{ m}^2/\text{s}
\]

(d) Period \( P = 2\pi \sqrt{\frac{r^3}{\mu}} \).

\[
P = 2\pi \sqrt{\frac{(6.978 \times 10^6)^3}{3.986 \times 10^{14}}} = 5801 \text{ s} = 96.7 \text{ min}
\]

Notice that the 600 km orbit has a larger energy (8.6% higher), a larger angular momentum (4.5% higher), a longer period (14.4% higher), and a lower orbital velocity (4.4% lower) than that of the Earth-surface orbit. Note that with these equations we can examine and compare the orbits of spacecraft in orbit around celestial bodies.

Now we consider a hypothetical LM ascent from the Moon and rendezvous at apolune (where the LM would be traveling the slowest). The LM elliptical trajectory’s radius to perilune must be greater than the radius of the Moon so that the LM does not intersect the surface of the Moon [11].

For our model, we need:

- Height of the orbiting CSM = \( h \)

- Moon Mass = \( M = 7.3477 \times 10^{22} \text{ kg} \)
3.1. THE ELLIPSE

- Moon Radius = \( r_m = 1.7371 \times 10^6 \text{ m} \)

- \( G = 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2} \)

- \( \mu_m = GM = 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2} \times 7.3477 \times 10^{22} \text{ kg} = 6.67 \times 10^{-11} \times 7.3477 \times 10^{22} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2} = 4.9009 \times 10^{12} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2} \)

- \( r_p = r_m = 1.7371 \times 10^6 \text{ m} \)

- \( r_a = r_m + h = 1.7371 \times 10^6 \text{ m} + h \)

- \( 2a = r_a + r_p = \frac{r_a + r_p}{2} \)

---

**Figure 3.2:** Lengths of \( r_a \) and \( r_p \) equal to \( 2a \).
First, we examine specific examples. We will assume the CSM is orbiting the Moon at an altitude of 60 nmi (1 nautical mile = 1,852 meters) and the LM is on the Moon. At this point, we only seek to find orbital characteristics.

**Example 3.1.3.**
For a CSM in a 60 nmi orbit above the Moon. We find:

(a) the orbital velocity,

(b) the energy,

(c) the angular momentum, and

(d) the period.

**Solution.** Since $h = 60 \text{ nmi} = 111.12 \text{ km} = 111,120 \text{ m} = 1.11 \times 10^5 \text{ m}$, $r = r_m + h = 1.7371 \times 10^6 + 1.11 \times 10^5 = 1.848 \times 10^6$

(a) $V_{CSM} = \sqrt{\frac{\mu_m}{r}} = \sqrt{\frac{4.9009 \times 10^{12}}{1.848 \times 10^6}} = 1,628.5 \text{ m/s} = 5,862.6 \text{ km/h}$

(b) $E_{CSM} = -\frac{\mu_m}{2r} = \frac{-4.9009 \times 10^{12}}{2(1.848 \times 10^6)} = -1.32 \times 10^6 \text{ m}^2/\text{s}^2$

(c) $L_{CSM} = rV \cos \phi = (1.848 \times 10^6) \times 1,628.5 = 3.01 \times 10^9 \text{ m}^2/\text{s}$

(d) $P_{CSM} = 2\pi \sqrt{\frac{r^3}{\mu_m}} = 2\pi \sqrt{\frac{(1.848 \times 10^6)^3}{4.9009 \times 10^{12}}} = 2\pi \sqrt{1.288 \times 10^6} = 7,130 \text{ s} = 118 \text{ mins}$

Now, for the LM, we examine the case where the length to perilune is equal to the radius of the Moon, $r_p = r_m$. 

---

30
Example 3.1.4.

For the LM in an elliptical trajectory around the Moon with \(r_p = r_m\) and \(r_a = r_m + h\), with \(h\) as in Example 3.1.3. We find:

(a) the energy,
(b) the angular momentum,
(c) the velocity at apolune,
(d) the period, and
(e) the launch velocity.

We have:

\[
\begin{align*}
  r_p &= r_m = 1.7371 \times 10^6 \text{ m} \\
  r_a &= r_m + h = 1.848 \times 10^6 \text{ m} \\
  2a &= r_a + r_p = 1.848 \times 10^6 + 1.7371 \times 10^6 = 3.585 \times 10^6 \text{ m}
\end{align*}
\]

and

\[
  a = \frac{r_a + r_p}{2} = 1.793 \times 10^6 \text{ m}
\]

Since \(r_p = a(1 + e)\),

\[
e = 1 - \frac{r_p}{a} = 1 - \frac{1.7371 \times 10^6}{1.793 \times 10^6} = .03117
\]

Solution. (a) \(E_{LM} = \frac{-\mu m}{2a} = \frac{-4.9009 \times 10^{12}}{2(1.793 \times 10^6)} = -1.367 \times 10^6 \text{ m}^2/\text{s}^2\)

(b) Since \(p = a(1 - e^2) = 1.793 \times 10^6(1 - .03117) = 1.74 \times 10^6\) and \(p = \frac{L^2}{\mu},\)
CHAPTER 3. ELLIPSE IN POLAR COORDINATES

\[ L_{LM} = \sqrt{p \mu_m} = \sqrt{(1.74 \times 10^6) \times (4.9009 \times 10^{12})} = 2.92 \times 10^9 \ m^2 / s \]

(c) \[ V_a = \frac{L}{r_a} = \frac{2.92 \times 10^9}{1.848 \times 10^6} = 1,580.09 \ m/s \]

(d) \[ P_{LM} = 2\pi \sqrt{\frac{\mu}{\mu_m}} = 2\pi \sqrt{\frac{(1.793 \times 10^6)^3}{4.9009 \times 10^{12}}} = 2\pi \sqrt{1.18 \times 10^6} = 6,814.16 \ s = 113.56 \ min \]

(e) \[ V_{LM} = \sqrt{2(E + \frac{\mu_m}{r_m})} = \sqrt{2(-1.367 \times 10^6 + \frac{4.9009 \times 10^{12}}{1.7371 \times 10^6})} = 1,705.47 \ m/s \]

Hence in order for the LM to reach a desired CSM orbit, the launch velocity will depend on the altitude of the CSM orbit.
4.1 Model for Potential Rendezvous

The assumptions in our lunar docking model are similar to the assumptions made in [11] and describe the restricted two-body problem:

1. The larger body (Moon) can be treated as a point with its attractive force concentrated at its center.

2. The smaller body (CSM/LM) can also be treated as a point mass.

3. The mass of the Moon is so much greater than the mass of the spacecraft that the center of mass of the system can be located at the center of mass of the Moon, and the attractive force of the spacecraft can be neglected.
4. The gravitational attraction of any other bodies can be ignored.

5. There are no other external forces other than gravitation acting on the spacecraft.

### 4.2 Potential Rendezvous

The two most interesting potential rendezvous points along an elliptical trajectory are at perilune and apolune. By Kepler’s second law, the rocket is moving the fastest at the perilune and slowest at apolune. Therefore, if the CSM is moving along an elliptical orbit around the Moon, the LM has to move to an elliptical orbit with the Moon as one of its foci and at least tangent to the CSM’s orbit. We examine potential rendezvous orbits and propose the optimal rendezvous orbits for the CSM and LM.

### 4.3 Elliptical Trajectories

In accordance with the laws of Kepler, we will assume in our model without loss of generality, that the orbits can be modeled by ellipses around the Moon
in a 2-D plane with the Moon as a focus located at the origin. Let

\[ F(x, y) = \frac{(x - c_1)^2}{A^2} + \frac{y^2}{B^2} - 1 = 0 \]

represent the CSM’s elliptical orbit around the Moon. \( F(x, y) \) is centered at \((c_1, 0)\), has length of major axis of \( A \), length of minor axis of \( B \), vertices at \((-A + c_1, 0)\) and \((A + c_1, 0)\), and foci at \((0, 0)\) and \((2c_1, 0)\). Values for \( A \), \( B \), and \( c_1 \) are known. Stating this equation for the orbiting CSM, we will build the model for the LM.

As previously stated, the CSM and LM are both moving fastest at perilune and slowest at apolune (Page 19). Hence, it would be less challenging for the LM to dock onto the CSM at apolune. Since the LM is also traveling along an elliptical orbit around the moon with the Moon as a focus, its slowest velocity is also achieved at apolune. Hence, the optimal rendezvous point for the LM to dock back onto the CSM is at apolune - when both are traveling the slowest and the orbits are tangent. See Figures 4.1 and 4.2 for a visual representation of the docking.

Interestingly enough, the LM’s trajectory to reach tangentially the CSM orbit apolune is not unique. In fact, there are infinitely many possible trajectories that the LM can take to dock back on to the CSM at apolune, as stated in the following theorem.
Theorem: Consider an ellipse. Then, there are infinitely many internal ellipses which have one common focus with the ellipse and are tangent at apolune (the point that is farthest from the shared focus).

Proof. An ellipse is a planar curve in which the sum of the distances from two foci to every point on the ellipse is constant. Without loss of generality, we can assume that the following equation describes our ellipse $E_0$.

$$E_0(x, y) = \frac{x^2}{a_0^2} + \frac{y^2}{b_0^2} - 1 = 0$$

The ellipse is centered at the origin and has $x$-intercepts at $(-a_0, 0)$ and $(a_0, 0)$. Its foci, $F_1$ and $F_2$, have coordinates $(-c, 0)$ and $(c, 0)$, respectively. $F_1$ and $F_2$ are separated by a distance of $2c$. Let $r_1^a$ and $r_2^a$ be the distances from $F_1$ and
4.3. ELLIPTICAL TRAJECTORIES

![Figure 4.3: General Ellipse](image1.png)

![Figure 4.4: $E_x(x, y)$ and $E_y(x, y)$](image2.png)

$F_\alpha$ to any point along the ellipse, respectively, where $\alpha$ is the angle between the $r_1^\alpha$ and the $x$-axis, see Figure 4.3.

Hence, for any point on the ellipse, the equation $r_1^\alpha + r_2^\alpha = 2a_0$ holds. Assume that there is an internal ellipse, $E_1$ with a common focus, $F_\alpha$ and tangent at $(a_0, 0)$ to $E_1$. Let $F_\beta$ be $E_1$'s other focus located on the $x$-axis (by symmetry) and $m = r_1^\beta + r_2^\beta$, where $\beta$ is the angle between $r_1^\beta$ and the $x$-axis. Since $E_1$ is an internal ellipse, see Figure 4.4, we have:

$$m_i = r_1^\beta + r_2^\beta < r_1^\alpha + r_2^\alpha = 2a_0$$

For $\beta = 0$,

$$m_i = r_1^0 + r_2^0 = 2r_2^0 + d(F_1, F_i)$$

Note that for any fixed $m_i$, $m_i < 2a_0$. For each choice of $r_2^0 \in \{0, r_2^0\}$, the equation

$$m_i = 2r_2^\beta + d(F_1, F_i)$$
has a unique solution for $F_i$. Therefore, there are infinitely many internal ellipses passing through $(a_0,0)$ with a common focus $F_1$. The smaller ellipses are of the form $E_i(x,y) = \frac{(x-x_0)^2}{a_i^2} + \frac{y^2}{b_i^2} - 1 = 0$, where $x_0$ is the shift along the $x$-axis. Since $E_i$ has $x$-intercept $(a_0,0)$, then

$$E_i(a_0,0) = \frac{(a_0-x_0)^2}{a_i^2} - 1 = 0$$

Hence,

$$\frac{(a_0-x_0)^2}{a_i^2} = 1$$

and

$$(a_0-x_0)^2 = a_i^2.$$ 

This gives a positive root as: $(a_0-x_0) = a_i$. So, $x_0 = a_0 - a_i$. These equations describe the shift, $x_0$ for each $E_i$. Also,

$$\frac{\partial E_i}{\partial x}(x,y) = \frac{2(x-x_0)}{a_i^2}$$

$$\frac{\partial E_i}{\partial x}(a_0,0) = \frac{2|a_0-(a_0-a_i)|}{a_i^2} = \frac{2a_i}{a_i^2} = \frac{2}{a_i} \neq 0$$

and

$$\frac{\partial E_i}{\partial y}(x,y) = \frac{2y}{b_i^2}$$

$$\frac{\partial E_i}{\partial y}(a_0,0) = 0$$
4.3. ELLIPTICAL TRAJECTORIES

Hence, the tangent line at the point \( (a, 0) \) is of the form \( x = \text{constant} = a_3 \). Hence all \( E_i \) are tangent to the original ellipse at apolune and there are infinitely many of these ellipses.

Now, out of the infinite possible trajectories, we want to find the optimal trajectory to reach and dock at apolune. Figure 4.5 depicts possible orbits to reach apolune.

![Figure 4.5: Potential Trajectories](image)

Note that if the CSM’s orbit is nearly circular, much like the orbits of planets around the Sun, then the travel distance for the LM would be the shortest. Figure 4.6 shows the CSM following a nearly circular trajectory and the LM following an elliptical trajectory.
Having the CSM maintain a nearly circular orbit not only minimizes the distance that the LM has to travel to reach and dock onto the CSM, but it also minimizes energy and fuel. The optimal orbit for the LM is one that is high enough to reach apolune and not be pulled back to the Moon. We still need to determine how close is the second focus to apolune. We start with discussing circular orbits.

4.4 The Copernican Model

In 1543, the Polish mathematician and astronomer, Nicolas Copernicus, published work proposing an astronomical model known as a heliocentric solar system. In his heliocentric model, he positioned the Sun as the center of the
universe with the Earth, and all the other planets, revolving around the Sun. Copernicus calculated that the planets traveled along circular orbits around the Sun, see Figure 4.7. At the time, this model caused quite a stir since previous beliefs and past models proposed were of an Earth centered solar system [9]. Even though Copernicus’ model was not entirely correct, he was not too far off.

The circular orbits that he proposed the planets traveled along were nearly circular. However, data collected from observations showed some errors in this description and he himself introduced various corrections to the equations of the orbits. Kepler improved the Copernican theory showing that plan-
ets travel along elliptical orbits around the Sun; these equations were confirmed by Tycho Brahe’s data and reconfirmed by later observations. Now we will discuss the actual elliptical orbits for planets in our solar system.

**Definition 4.4.1.** *The orbital eccentricity*, $e = c/a$ (where $c$ is the length from the center of the ellipse to a focus and $a$ is the length of the semimajor axis of the elliptical orbit, see Figure 4.3), determines the amount by which an orbit deviates from a perfect circle.

Table 4.2 shows the eccentricity, $e$, for quadratic curves (conic sections) that can represent various trajectories of traveling objects. Note: when $e$ is closer to 0 than 1, the orbit of a planet will be more circular. Table 4.1 shows actual eccentricity for each of the planets in the solar system. Note that all orbits are almost circular with Mercury’s orbit being the most elliptical. Thus,
4.4. THE COPERNICAN MODEL

circles minimize energy of trajectories in the solar system and non-zero eccentricity can be explained by the influence of other masses in the model.

With that observation we now develop a hypothetical Moon landing mission.
5.1 Hypothetical Mission

Now, with our assumptions in tact, see Section 4.1, we describe the Lunar Orbit Rendezvous (LOR) flight following NASA’s plan. We first need to leave Earth. To reach the Moon, we need to equip a rocket with enough fuel, speed, and force. The rocket must first leave Earth’s gravity. When the rocket is on a high enough orbit, it must travel to the Moon. Once the rocket is on the Moon’s orbit, the LM is sent to the Moon’s surface. Then the crew carry out any lunar explorations on the lunar surface and when done, they prepare the LM to leave the Moon’s gravity [13]. (Also, see our assumptions in Section 4.1.)
5.1. HYPOTHETICAL MISSION

It too needs enough fuel, speed, and force to leave the Moon's gravity. Once the LM reaches the desired orbit, it must reach the orbiting CSM with a specific velocity and dock at a very specific moment to succeed. Once docked, the CSM can leave the Moon's orbit and return to Earth. Figure 5.1 outlines each stage using the Saturn V Rocket as an example [1].

Since the 20th century, humans have been able to send objects of various functions and sizes into space (such as satellites, the Hubble Space Telescope, and the Mars rover). Whereas, getting objects back from celestial bodies is a much more challenging task and there were just several missions that were
successful. Our main focus is the docking of the LM back onto the orbiting CSM, see stages 13-15 in Figure 5.1. We also calculate the optimal orbit(s) and rendezvous point(s) for the CSM and LM.

For a successful rendezvous and docking between a maneuverable vehicle (LM), which would perform all of the maneuvers, and its target (CSM) requires that velocity, time, and position of the vehicles be matched simultaneously [14]. In theory, it would be best to launch the LM in a vertical direction. Yet, this is not ideal as the LM would intersect the CSM’s orbit at a high velocity transversally and would have problems with docking and not have enough time to perform complicated turning maneuvers. Hence, we seek alternate trajectories tangent to the CSM’s orbit.

### 5.2 Match of Velocities

Assuming that the CSM maintains a circular lunar orbit, the orbital velocity would be given by $V_{CSM} = \sqrt{\frac{\mu_m}{r}}$, where $\mu_m$ is equal to the product of $G$ and the mass of the Moon and $r$ is the radius of the Moon plus the altitude of the spacecraft to the surface of the Moon (see Page 25). Here we also assume that the initial orbit of the LM is an ellipse and the docking point is at apolune, see
5.2. MATCH OF VELOCITIES

Figure 5.2. Hence, the velocity of the LM is given by: \( V_{LM} = \sqrt{2(E + \frac{\mu_m}{r})} \).

To match velocity of the two spacecraft we need:

\( V_{CSM} = V_{LM} \).

Substituting for the orbital velocity of the CSM and the orbital velocity of the LM:

\[
\sqrt{\frac{\mu_m}{r}} = \sqrt{2(E + \frac{\mu_m}{r})}.
\]

Squaring both sides, we get:

\[
\frac{\mu_m}{r} = 2(E + \frac{\mu_m}{r}).
\]
Solving for the energy, $E$:

\[
\frac{\mu_m}{r} = 2E + 2 \frac{\mu_m}{r}
\]

\[
\frac{\mu_m}{r} - 2 \frac{\mu_m}{r} = 2E
\]

\[
\frac{\mu_m}{r} - 2 \frac{\mu_m}{r} = 2E
\]

\[
-\frac{\mu_m}{r} = 2E
\]

\[
-\frac{\mu_m}{2r} = E
\]

Notice that the energy of the LM orbit is equal to the energy of the CSM’s circular orbit. Hence, for the velocities to match, the orbit of the LM should also be circular and almost exactly equal to the CSM’s orbit. We proved the following theorem:

**Theorem:** The optimal rendezvous orbits of two objects in space are both equal and circular.

Hence, during docking the two spacecraft will be moving parallel to each other. Since the circular orbits are the lowest energy orbits, this solution should require the minimal amount of fuel.
5.3 Launch Angle

To achieve the desired orbit, the LM initially should take off from the Moon in a vertical ascent, but quickly tilt (pitchover) and further adjust its trajectory angle by using gravity. The gravity turn is an orbital maneuver used to launch spacecraft into orbit around a celestial body [15]. The initial pitchover maneuver, at the desired angle, uses gravity to turn downward and puts the spacecraft on the correct trajectory to achieve the desired orbit, see Figure 5.3. However, the equations for the gravity turn trajectory cannot be solved analytically for an object changing its mass and must be calculated numerically.

![Figure 5.3: Gravity Turn](image-url)
This means the calculations estimate the best flight paths and the craft’s trajectory may have to be adjusted during the flight. In the next section we show how to approximate the paths.

5.3.1 Mathematical Model for a Gravity Turn

Consider a point mass vehicle with a constant mass $m$ in a uniform gravitational field (neglecting air resistance). The differential equation of motion of the vehicle is given by:

$$m \frac{d \vec{v}}{dt} = \vec{F} - mg \hat{k}$$

where $\vec{F}$ is the thrust force vector (a function of $t$), $\vec{v}$ is the velocity vector (a function of $t$), and $\hat{k}$ is a unit vector in the vertical direction, $m$ is the instantaneous mass of the vehicle (also a function of $t$ during the flight).

By separating components parallel to $\vec{v}$ and those perpendicular to $\vec{v}$, we get the system:

$$\frac{dv}{dt} = g \left( \frac{F}{mg} - \cos \beta \right)$$

$$v \frac{d \beta}{dt} = g \sin \beta$$

By dividing both sides, we have:

$$\frac{1}{v} \frac{dv}{d \beta} = \frac{g \left( \frac{F}{mg} - \cos \beta \right)}{g \sin \beta}$$
5.3. LAUNCH ANGLE

\[
\frac{dv}{d\beta} = \frac{v\left(\frac{F}{mg} \cos \beta\right)}{\sin \beta}
\]

Dividing by \(v\) and simplifying:

\[
\frac{1}{v} \frac{dv}{d\beta} = \frac{\frac{F}{mg} \cos \beta}{\sin \beta} \frac{d\beta}{d\beta}
\]

\[
\frac{d(\ln v)}{d\beta} = \frac{\frac{F}{mg} \cos \beta}{\sin \beta}
\]

Integrating with respect to \(\beta\):

\[
\ln v = \frac{F}{mg} \int \frac{1}{\sin \beta} \, d\beta - \int \frac{\cos \beta}{\sin \beta} \, d\beta
\]

Finally, we get:

\[
\ln v = \ln \left(\sin \frac{\beta}{2} + \frac{F}{mg} \left(- \ln \left(\cos \frac{\beta}{2}\right) + \ln \left(\sin \frac{\beta}{2}\right)\right)\right).
\]

This gives an explicit solution for the velocity along the gravity turn trajectory when the mass is constant. Unfortunately, the mass of our spacecraft is not constant as the rocket burns fuel, there is a corresponding fuel exhaust and also the used stages are detached and discarded. Thus, the model does not apply in general. However, we can use this solution to approximate the velocity locally for very small time intervals while assuming constant mass equal to the average mass during the time interval. Hence, we need to estimate the mass and velocity at various times along the trajectory.
5.3.2 Tsiolkovsky Rocket Equation and Propellant Requirements

In 1903, Konstantin Tsiolkovsky, a Polish-Russian rocket scientist, determined the dynamics that explained the motion of rockets in his work that is now considered a foundation of astronautics: Tsiolkovsky’s formula [16]. His rocket equation 5.1, relates mass to velocity changes [15].

\[ \Delta v = v_e \ln \left( \frac{m_0}{m_f} \right) \]  \hspace{1cm} (5.1)

Here \( m_0 \) is the initial mass of the rocket before the expulsion of propellant, \( m_f \) is the final mass of the rocket after the expulsion of propellant, and \( v_e \) is the exhaust velocity of the propellant. The exhaust velocity of the propellant is given by the equation:

\[ v_e = I_{sp} g_0 \]  \hspace{1cm} (5.2)

In Equation 5.2, \( I_{sp} \) is the specific impulse of the craft in seconds and is a measure of the rocket’s propellant efficiency (the amount of thrust per propellant weight flow rate depends on the fuel used) and \( g_0 \) is the acceleration at the Earth’s surface equal to 9.8 m/s². Hence we can derive the equation that yields the propellant mass fraction of the whole rocket’s mass required to increase the velocity by \( \Delta v \). Taking Tsiolkovsky’s rocket equation and dividing
5.3. LAUNCH ANGLE

by the exhaust velocity, we get:

\[ \frac{\Delta v}{v_e} = \ln \left( \frac{m_0}{m_f} \right) \]

Exponentiating both sides, we have:

\[ e^{\frac{\Delta v}{v_e}} = \frac{m_0}{m_f} \]

\[ (e^{\frac{\Delta v}{v_e}})^{-1} = \left( \frac{m_0}{m_f} \right)^{-1} \]

\[ e^{-\frac{\Delta v}{v_e}} = \frac{m_f}{m_0} \]

Since \( m_0 \) is the mass of the rocket (\( m_f \)) plus the mass of the propellant (\( m_p \)), we replace \( m_0 = m_f + m_p \) in the denominator and add and subtract the mass of the propellant in the numerator to get:

\[ e^{-\frac{\Delta v}{v_e}} = \frac{m_f}{m_0} = \frac{m_f + m_p - m_p}{m_f + m_p} \]

Simplifying we get the mass fraction of the whole rocket’s mass:

\[ e^{-\frac{\Delta v}{v_e}} = 1 - \frac{m_p}{m_f + m_p} \]

\[ -1 + e^{-\frac{\Delta v}{v_e}} = -\frac{m_p}{m_f + m_p} \]

\[ 1 - e^{-\frac{\Delta v}{v_e}} = \frac{m_p}{m_f + m_p} \]

\[ 1 - e^{-\frac{\Delta v}{v_e}} = \frac{m_p}{m_f + m_p} = M_f \]
\[ M_f = 1 - e^{-\frac{\Delta v}{v_e}} \]  \hspace{1cm} (5.3)

Where \( M_f \) is the fraction of initial mass representing required propellant. Hence, if we know the exhaust velocity, \( v_e \), and the \( \Delta v \), and the weight of the payload, we can calculate the amount of propellant necessary. As an example, we calculate the necessary amount of propellant in some cases.

**Example 5.3.1.**
Let the dry mass of the LM ascent stage be 4,796 pounds (2,180 kg) and assume an exhaust velocity of 4,500 m/s and \( \Delta v \) of 9,700 m/s. We calculate \( M_f \) and the total propellant necessary.

**Solution.** Using a single stage vehicle,

\[ M_f = 1 - e^{-\frac{\Delta v}{v_e}} = 1 - e^{-\frac{9,700}{4,500}} = 1 - 0.116 = 0.884 \]

Hence in this case, about 88.4\% of the initial mass must be propellant and the rest is the engines, tanks, and payload of the LM. So, the dry mass of the LM ascent stage was 4,796 pounds, constitutes only 11.6\% of the total mass. \( \square \)

Next we calculate and redo Example 5.3.1 assuming the LM makes use of two stages.

**Example 5.3.2.**
Again, assume the dry mass of the LM ascent stage is 4,796 pounds (2,180 kg) and an exhaust velocity of 4,500 m/s. Now assume the Δv of the first stage is 5,000 m/s and the Δv of the second stage is 4,700 m/s. We calculate $M_f$ for the first stage, the second stage, and the total propellant necessary.

**Solution.** Using a two stage vehicle, the mass fraction of the first stage is:

$$M_{f1} = 1 - e^{-\frac{\Delta v}{v_e}} = 1 - e^{-\frac{5000}{4500}} = 0.671$$

For the first stage, about 67.1% of the initial mass must be propellant. After disposing of the first stage, the mass is equal to 32.9% minus the mass of the tank and engines of the first stage. Assume that this is 8% of the initial total mass, then only 24.9% remains.

For the second stage:

$$M_{f2} = 1 - e^{-\frac{\Delta v}{v_e}} = 1 - e^{-\frac{4700}{4500}} = 0.648$$

Hence, 64.8% of the remaining mass must be propellant (16.2% of the initial mass), and 8.7% remains for the payload, tank, and engines of the second stage. So, 16.7% is available for the payload and the rest of the ship. □

Notice that by using two stages, this allows the spacecraft to bring 5.1% more mass into space. Hence, by using multiple stages, we can increase the
mass of the LM ascent stage. Next we calculate the necessary amount of propellant for a LM under different circumstances. We use the LM’s actual weight, exhaust velocity, and $\Delta v$ (launch velocity) from our calculations in Example 3.1.4.

**Example 5.3.3.**
Let the dry mass of the LM ascent stage be 4,796 pounds (2,180 kg) and assume an exhaust velocity of 3,050 m/s and $\Delta v$ of 1,705.47 m/s. We calculate $M_f$ and the total propellant necessary.

**Solution.** Again, using a single stage vehicle, the mass fraction is:

$$M_f = 1 - e^{-\frac{\Delta v}{v_e}} = 1 - e^{-\frac{1,705.47}{3,050}} = 1 - 0.572 = 0.43$$

Hence in this case, about 43% of the initial mass must be propellant and the rest is the engines, tanks and payload of the LM. So, the dry mass of the LM ascent stage was (4,796 pounds) was 57% of the total mass. 

Next we recalculate the mass of the propellant for the known $\Delta v$ of the LM at ascent of 1,847.09 m/s [17].
5.3. LAUNCH ANGLE

Example 5.3.4.
Let the dry mass of the LM ascent stage be 4,796 pounds (2,180 kg) and assume an exhaust velocity of 3,050 m/s and $\Delta v$ of 1,847.09 m/s. We calculate $M_f$ and the total propellant necessary.

Solution. Again, using a single stage vehicle, the mass fraction is:

$$M_f = 1 - e^{-\frac{\Delta v}{v_e}} = 1 - e^{-\frac{1,847.09}{3,050}} = 1 - 0.546 = 0.454$$

Hence in this case, about 45.4% of the initial mass must be propellant and the rest is the engines, tanks and payload of the LM. So, the dry mass of the LM ascent stage was (4,796 pounds) was 54.6% of the total mass. □

Notice that to achieve a higher $\Delta v$, the spacecraft needs a significantly larger amount of fuel. Fortunately, using multiple stages (and discarding additional mass), allows us to send larger spacecraft into space. Note that the actual LM was a single stage vehicle. See also [18] and [19] for further details on the Apollo 11 LM propellant use.

5.3.3 Launch Time

Having the calculated trajectory, right amount of fuel to follow the trajectory to achieve the orbit, the LM needs to be launched at the right time to be at the
assigned meeting point with the CSM. The rendezvous and docking requires timing to be very precise [20].

**Figure 5.4:** Timing a rendezvous and docking. The CSM passes the potential rendezvous point, $p$. The LM launches 33 minutes after the CSM passes $p$. It takes the LM 85 minutes to reach $p$.

We know that the period of the CSM in a circular lunar orbit at an altitude of 111,120 meters makes one complete revolution in 118 minutes, see Example 3.1.3. This is found with the equation:

$$P = 2\pi \sqrt{\frac{r^3}{\mu_m}}$$
5.3. Launch Angle

First, assume it takes the CSM $T$ minutes to complete an entire revolution and it takes the LM $t$ minutes to reach the desired CSM orbit at the assigned point, $p$, where desire to dock.

We can estimate $t$ from the gravity turn. This can be done by hand if calculations are done for example for every second. To rendezvous and dock at $p$, the LM must launch $T - t$ minutes after the CSM passes point $p$, right before the proposed launch.

Figure 5.4 shows a rendezvous and docking in which the LM takes 1 hour and 25 minutes to reach point $p$ at the CSM orbit. Hence, the LM would launch 33 minutes after the CSM passed the meeting point right before the liftoff. Note that $p$ may not be visible from the launch site and some additional easy calculations may have to be done to estimate the right launch time.
CONCLUSIONS

We began modeling the CSM and LM rendezvous and docking orbits using ellipses to decide if this is computationally feasible to perform. Assuming that the CSM maintains an elliptical orbit around the Moon, with the Moon as one of its foci, we decided that the LM’s orbit, should have a common foci (the Moon) and should at least be tangent to the CSM’s orbit at one point, at apolune. Interestingly enough, there are infinitely many ellipses that have a shared focus and are tangent at apolune.

When we studied planets’ orbital eccentricity we realized that their orbits around the Sun are nearly circular. Energy calculations showed that the CSM should maintain a low energy circular orbit, this would shorten the travel dis-
tance for the LM as well. The CSM would also have constant velocity along the circular orbit. Since if the LM’s trajectory is elliptical, the CSM and LM velocities would not match, as they would be on different orbits. Therefore, for a successful rendezvous and docking, position, time, and velocity, of the CSM and LM orbits must match, hence the LM has to reach and match the CSM’s circular orbit.

Next we presented numerical solutions for the launch angle, gravity turn, propellant requirements, and timing. We conclude that the calculations during the space missions in the 20th century have been tedious, but not impossible. Currently, NASA has made many resources available to the public that detail all missions: logs, transcripts, press kits, journals, and other publications containing detailed information and thorough calculations.

Our model is in agreement with published sources. Therefore we believe that calculations and official models at the time were sufficient of the success of the missions (in addition to many technological challenges). Modern computers use similar underlying principles for fast trajectory determinations and systematic updates. Space station missions showed that small spacecraft can easily dock when the station maintains a circular orbit around the Earth. While Moon missions are not continued, it looks like docking, while com-
plex and difficult, is possible and may not be a greatest obstacle for further space exploration. Hence the doubts about the 1969 mission complexity are groundless.
BIBLIOGRAPHY


