A broad review of important wave equations, solutions, applications, and numerical modeling.

By

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A broad review of important wave equations, solutions, applications and numerical modeling.

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3 to 5 keywords or phrases to describe the item

Matthew Costa

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Dedication

To my mom, the youngest of six. Who decided to be the only one in her family to go to college and study Computer science in the 70's. Who would come home after work and spend time reading books and doing puzzles with me. Who always held me to a high standard.

To Maryann Berguson elementary school, Cabrillo college, and CSU Channel Islands for creating an educational environment where knowledge comes first.

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To the ocean, my inspiration for studying wave equations.

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Mo, Sharon and the Costanoa crew for setting me up with a great job so I could afford grad school.

To both of my parents who set me up for success.

Camarillo, California

May 6, 2019

ABSTRACT

Derivations of wave equations, various presentations of their solutions and MATLAB models are presented. Thereafter, basic ocean wave forecasting will be discussed along with it's applications.

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CHAPTER 1

INTRODUCTION

Inspiration for this thesis comes from the authors recreational love of the ocean and strong desire to understand mathematical principles that govern ocean waves.

Society's current ability to model the ocean has limitations. Although mathematical physicists have made much progress in these fields theoretically, things become different when we start implementing theoretical solutions to the deep blue sea.

There are a seemingly infinite amount of parameters, some we can determine, and some that we cannot. Even the ones that we can control, like wind speed are based off of approximations, thus weakens existing models. In addition to infinite amounts of parameters the computers we use have limitations themselves. Although computers can store very large numbers, precise analytical solutions require a true infinite and continuous domain. Unfortunately, computers are not there yet.

To rigorously study the types of ocean waves that the author is interested in, it would require a Phd dissertation due to the complexity and collaboration amongst all types of professionals. Therefore, the aim of this thesis is to get our toes wet and get a feel for the fundamental ideas behind wave motion. What better way to start procuring this field than with the wave equation. The wave equation is a fundamental partial differential equation that describes surface water waves, sound, light and seismic waves. Knowledge of these waves is used in such fields as acoustics, electromagnetic, and fluid dynamics. The wave equation has spatial and time variables and will be used in the one and two dimensional forms. Explicit definitions will be stated in section 2.3.

Wave equations should be of interest to all beings of planet earth as many forces are governed by the wave equation. Such forces include tsunamis, ocean storms, and even the microwaves that warm up TV dinners! Weather research is focused towards understanding storms that could potentially wipe us off the face of the planet or for recreational use such as surfing. Wave equations along with their solutions and models should clearly be an important area of study.

This thesis is organized as follows. Chapter 2 will be dedicated to important definitions and theorems that lay the groundwork for the analytical and numerical solutions presented in chapter 3 and chapter 4 respectively. Chapter 2 and 3 may be of interest to partial differential equation students who are learning about basic PDE's. Chapter 2 and 3 will serve as a great fundamental step for understanding notation and derivations for other PDE's. Upon introducing analytic descriptions of solutions to the wave equation in Chapter 3 we quickly see that there are limitations to real-life applications. For these reasons, Chapter 4 is dedicated to deriving a numerical scheme that approximates solutions to the wave equation. Those who wish to find a thorough explanation of how to discretize a differential equation chapter 4 will serve as a great guide. Chapter 5 will contain snippets from the 1-D and 2-D code presented in chapter 4. We hope that ocean lovers find chapter 6 especially exciting as the wave function is introduced to model deep water ocean swells, shallow water wave equations are derived and the Navier-Stokes equations are presented. Finally, in chapter 7 conclusions are made, future research ideas are presented along with applications. My main contributions to this thesis include adoptive numerical implementation of 1 and 2 dimensional wave numerical solvers to model wave interaction with Dirichlet and reflecting boundary conditions. This thesis is written thoroughly involving deep discussion on current real world applications in chapter 6 and 7. Most of which are focused around ocean waves.

Upon finishing this thesis the reader should feel comfortable with what the wave equation is, where it comes from, and its different forms. The reader should have an understanding of the methods of deriving the very long analytical solutions and become accustomed with a typical numerical scheme to implement partial differential equations into computer programs such as MATLAB. Last and most importantly, the reader should look at life with eyes wide open, even more than before. The reader should excitingly notice the wave equation in everyday life and even better yet come up with ideas for which the wave equation has not yet been applied.

The following notation will be used throughout this thesis. For the sake of simplicity, the author has chosen to use subscripts to denote partial differentiation,

$$\frac{\partial}{\partial t}\mu(x,t) = \mu_t(x,t) = \mu_t,$$

where the dependence on (x, t) will be omitted when it is clear from the context.

CHAPTER 2

Preliminaries and Definitions

In this section of the thesis we will recall essential definitions used throughout this manuscript. The reader is especially encouraged to understand Theorems 2.1.2 and 2.1.3. Taylor series are the backbone of the numerical methods of the thesis while Fourier series are the backbone of the analytical section of the thesis.

Although explicit definitions are given below it is exciting to remind the reader that numerical section uses polynomials to approximate its solutions while the analytic section uses sinusoidal functions to approximate its solutions!

2.1 Definitions

In section 2.1 we state definitions that serve as the backbone to our theoretical knowledge of wave equations. They are fundamental definitions that are extremely important to any field of analysis.

Definition 2.1.1 *The Taylor series for a given function $f(x)$ centered at a is defined as*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+$$

whenever it converges in the appropriate metric.

Theorem 2.1.2 (Taylor's Theorem) *Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x_0)$ between x_0 and x with*

$$f(x) = P_n(x) + R_n(x),$$

where

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x-x_0)^k, \end{aligned}$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}.$$

Definition 2.1.3 *Fourier series*

The Fourier series for a given function f is defined as

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n\cos(nx) + B_n\sin(nx)) \tag{†}$$

with coefficients

$$A_n = \int_{-\pi}^{\pi} f(y)\cos(ny)\frac{dy}{\pi}, \quad \textit{for} \quad n \in \mathbb{N},$$

$$B_n = \int_{-\pi}^{\pi} f(y)\sin(ny)\frac{dy}{\pi}, \quad \textit{for} \quad n \in \mathbb{N},$$

Thus the N-th partial sum of the series is

$$S_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N}(A_n\cos(nx) + B_n\sin(nx)).$$

Remark 2.1.4 *The series (†) is defined whenever $S_N(x) \longrightarrow$ (†) as $N \to \infty$ in the appropriate metric.*

Remark 2.1.5 *We will begin to introduce a framework for analyzing differential operators with a special focus on the wave equation posed on a bounded interval of length l.*

$$\begin{aligned} &u_{tt} = c^2\Delta u, \quad x \in [0, l) \\ &u(x,0) = \phi(x) \qquad\qquad . \\ &u_t(x,0) = \Psi(x) \end{aligned} \tag{2.1.1}$$

In equation (2.1.1), we used Δ to denote u_{xx}. In the 2-D setting Δ is used to denote $u_{xx}+u_{yy}$. In general, in $\mathbb{R}^k$ with $x_1, x_2, ..., x_k$ variables $\Delta = \sum_{i=1}^{k} u_{x_{ii}}$. Further in the thesis, we will also study solutions to equation (2.1.1) in unbounded domains.

In order to understand solution to equation (2.1) we study "spectral properties" of the operator Δ.

Definition 2.1.6 *EigenValues and eigenfunctions of Δ*

The numbers $\lambda_n = (\frac{n\pi}{l})^2$ are called eigenvalues and the functions $X_n(x) = \sin(\frac{n\pi x}{l})$ are called eigenfunctions related to Δ.

Remark 2.1.7 *In chapter 3 we find eigenvalues and eigenfunctions for Δ in the 2-D setting.*

Theorem 2.1.8 *The Mean Value Theorem.*

Let f be a function that satisfies the following hypotheses:

1. *f is continuous on the closed interval $[a, b]$.*

2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b-a).$$

Since we consider series forms g solutions to the wave equation, we can introduce an appropriate metric space equipped with an inner-product.

Definition 2.1.9 *An inner product space is a vector space X with an inner product defined on X. An inner product on X is a mapping of* $X \times X$ *into the scalar field K of X; that is, with every pair of vectors x and y there is associated a scalar which is written* $\langle x, y\rangle$*, and is called the inner product of x and y, such that for all vectors x,y,z and scalars* α *we have*

$$(1) \qquad \langle x+y, z\rangle = \langle x, x\rangle + \langle y, z\rangle$$

$$(2) \qquad \langle \alpha x, y\rangle = \alpha\langle x, y\rangle$$

$$(3) \qquad \langle x, y\rangle = \overline{\langle y, x\rangle}$$

$$(4) \qquad \langle x, x\rangle \geq 0$$

$$(5) \qquad \langle x, x\rangle = 0 \iff x = 0.$$

Definition 2.1.10 *The $\mathbb{L}^2(R)$ space is the space of square-integrable functions for which the integral of the square of the absolute value is finite. Therefore, if*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty,$$

then f is in the $\mathbb{L}^2$ space. This definition can also be adapted to $f \in L^2[a,b]$.

In section 2.2 two different methods of determining how to classify a PDE as elliptic, hyperbolic, and parabolic. The reader is encouraged to pay special attention to the hyperbolic definitions as the wave equation is hyperbolic.

2.2 Classifying PDEs

Definition 2.2.1 *Elliptic, Hyperbolic, and Parabolic in terms of coefficients: Consider the following PDE, a linear equation of order two in two variables with six real constants.*

$$a_{11}u_{xx} + a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = 0$$

☺ *will denote terms of order* 1 *or* 0 *below in (i),(ii) and (iii).*

(i) Elliptic: If $a_{12}^2 < a_{11}a_{22}$ then it is reducible to $u_{xx} + u_{yy} + ☺ = 0$

(ii) Hyperbolic: If $a_{12}^2 > a_{11}a_{22}$ then it is reducible to $u_{xx} - u_{yy} + ☺ = 0$

(iii) Parabolic: If $a_{12}^2 = a_{11}a_{22}$ *then it is reducible to* $u_{xx} + ☺ = 0$, *unless* $a_{11} = a_{22} = a_{12} = 0$

From these definitions it is easy to see that the wave equation is Hyperbolic. Observe that the wave equation coefficients are the following $a_0 = a_1 = a_2 = a_{12} = 0$, $a_{22} = 1$, and $a_{11} = -(c^2)$. We see that the coefficients agree nicely with (ii). That is $0^2 > -(c^2)$.

Define coefficients of a PDE, form the matrix, recall that the product of the eigenvalues of a matrix is equal to the determinant of a matrix. Notice that we are working in the $n = 2$ case.

Definition 2.2.2 *Elliptic, Hyperbolic, and Parabolic PDE's in terms of diagonals of matrix:*

Take a matrix A, the coefficients of the PDE. Then, let D be the matrix of real numbers $d_1, ..., d_n$ *consisting of the eigenvalues of A. Apply a change of scale so that all the the* $D_{n's}$ *equal to* $+1, -1$, *or* 0.

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

(i)The PDE is called elliptic if all the eigenvalues $d_1, ..., d_n$ *are positive or all are*

negative. That is, $|d_1 + \cdots + d_n| = n$.

(ii)The PDE is called hyperbolic if none of the $d_1, ..., d_n$ *vanish and one of them has the opposite sign from the* $(n-1)$ *others. That is,* $\exists_i : \quad |d_1 + \cdots + d_n - 2d_i| = n$.

(iii)The PDE is called Parabolic if exactly one of the eigenvalues is zero and all the others have the same sign. That is, $|d_1 + \cdots + d_n| = n - 1$.

2.3 The Wave equation

In this section we will continue to introduce the wave equation and accompanying definitions. In order to understand the Matlab code presented in Chapter 4 it is crucial to well understand all definitions in this section.

Definition 2.3.1 *Basic wave equation*

$$u_{tt} = c^2 u_{xx} \quad for \quad -\infty < x < \infty$$

Definition 2.3.2 *Two Dimensional wave equation*

$$u_{tt} = c^2 (u_{xx} + u_{yy}) \quad for -\infty < x, y < \infty$$

Definition 2.3.3 *Dirichlet Boundary Space Conditions*

For the 1-D wave equation $u_{tt} = c^2 u_{xx}$ $0 < x < L,\ t > 0$ *and initial conditions* $u(x,0) = f(x),\ u_t(x,0) = g(x), \quad 0 < x < L$. *Since* $u(0,t) = u(L,t) = 0$

Definition 2.3.4 *Dirichlet boundary conditions (fixed boundary condition)*

For all (x, y) *on* S*, the fixed boundary of a region* R *we have that* $u(x, y)|_{(x,y)\in S} = g(x, y)$

Definition 2.3.5 *Neumann Boundary Conditions*

In the homogeneous case, $u_x(0, t) = 0$ *Say something about normal to the boundary of the domain being prescribed as data.*

The following definition will be used in chapter 4 for numerical implementation. Grid lines and mesh points will be used interchangeably through the thesis.

Definition 2.3.6 *Grid lines*

In a two dimensional system we can construct a grid by choosing integers m *and* n*. Let* $y \in [c, d]$ *with* $y_0 = c < y_1 < y_2 < \ldots < y_m = d$*. Similarly, let* $x \in [a, b]$ *with* $x_0 = a < x_1 < x_2 < \ldots < x_n = b$

Then we will obtain the width h *as* $h = \frac{b-a}{n}$ *and the width* k *as* $\frac{d-c}{m}$ *Letting* $i = 0, 1, .., n$ *and* $j = 0, 1, \ldots, m$ *we can obtain gridlines* $x = x_i = a + ih$ *and* $y = y_j = c + jk$*.*

Definition 2.3.7 *Mesh Points*

Mesh points are the intersections of grid lines. The points (x_i, y_i) may also be called nodes.

To define mesh points for the one dimensional wave equation consider

$$\frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0.$$

With $0 < x < l$, $t > 0$ and $u(0,t) = u(l,t) = 0$ for $t > 0$, $u(x,0) = f(x)$, and $\frac{\partial u}{\partial t}(x,0) = g(x)$, for $0 \le x \le l$.

Take integers m, k, i, j then the grid points will be $x_i = ih$ and $t_j = jk$ with $h = \frac{l}{m}$

Thus the wave equation at any one of these mesh points becomes

$$\frac{\partial^2 u}{\partial t^2}(x_i,t_j) - c^2 \frac{\partial^2 u}{\partial x^2}(x_i,t_j) = 0.$$

Definition 2.3.8 *Courant-Friedrichs-Lewy condition (CFL)*

The CFL will be defined as $\lambda = \frac{\alpha k}{h}$. With $\alpha, k, h \in \mathbb{R}$.

Along with definition 2.3.6 it should also be noted that we can rewrite the CFL as

$$\lambda = \alpha \frac{(d-c)n}{(b-a)m} = \alpha \frac{\Delta t}{\Delta x}.$$

Definition 2.3.9 *Forward and backward difference formulas*

Consider the difference quotient, i.e the classical definition of derivative as

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0)$$

If $h>0$ the formula is known as the forward-difference formula.

If $h<0$ then the formula is known as the backward-difference formula, for appropriate $f'(x_0)$.

The centered-difference formula will be distinguished as $f'(x_0)\approx\frac{f(x+h)-f(x-h)}{2h}$

Remark 2.3.10 *Newton's second law*

$$\sum \textit{Forces} = \textit{mass} \times \textit{acceleration}$$

Definition 2.3.11 *Below is the formal derivation of a numerical scheme to approximate a second order derivative. This will be used to derive 1-D and 2-D wave equations and is done by using Taylor series and difference formulas.*

We begin by consideration by derivation for f' and f'' using Taylor series.

Evaluate $f(x_0+h)$ and $f(x_0-h)$ at x_0, we obtain.

$$f(x_0+h)=f(x_0)+f'(x_0)(x_0+h-x_0)+\frac{f''(x_0)}{2!}h^2+\cdots+\frac{f^{n+1}(\zeta)}{(n+1)!}h^{n+1} \quad (2.3.1)$$

$$f(x_0-h)=f(x_0)+f'(x_0)(x_0-h-x_0)+\frac{f''(x_0)}{2!}(-h)^2+\cdots+\frac{f^{n+1}(\zeta)}{(n+1)!}(-h)^{n+1} \quad (2.3.2)$$

Adding and subtracting the function at these values shows us two nice things. A first and second derivative.

1. *Subtracting (2.3.1) from (2.3.2) we obtain* $f(x_0+h)-f(x_0-h) = f'(x_0)2h \pm \mathcal{O}(h^3)$

   Thus we can see that $f'(x_0) = \frac{f(x_0+h)-f(x_0-h)}{2h} \pm \mathcal{O}(h^2)$ *(the reader may also notice that this is a centered difference formula.*

2. *Adding (2.3.1) and (2.3.2) we obtain* $f(x_0+h)+f(x_0-h) = 2f(x_0)+f''(x_0)h^2 + \cdots + \mathcal{O}(h^4)$

   Thus we can see that

$$f''(x_0) = \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2} + \mathcal{O}(h^2) \tag{2.3.3}$$

Definition 2.3.12 *Discretization*

Using equation (2.3.3) we can conveniently rearrange a second order PDE to find the next "step". In equation (2.3.3) we can consider "h" as the step size. Note that we can consider h *as a forward step and* $-h$ *as a backward step. We are most interested in the forward step in the* x, y *and* z *axis. For the function* $u(x,t,y)$ *We will use subscripts* i, j, k *to denote the step of the function. For example,* $u(x_i, t_j, y_k)$ *will denote the current step of the function* u*. While*

$u(x_i, t_{j+1}, y_k)$ will denote the next step of function u in regards to the variable t that we will use for time.

Consider the second derivative of function u with respect to x,

$$u_{xx}(x_i, t_j) = \frac{u(x_{i+1}, t_j) + u(x_{i-1}, t_j) - 2u(x_i, t_j)}{\Delta x^2}.$$

We can rearrange this equation to solve for the next space step $u(x_{i+1}, t_j)$,

$$u(x_{i+1}, t_j) = u_{xx}(x_i, t_j)\Delta x^2 - u(x_{i-1}, t_j) + 2u(x_i, t_j).$$

Similarly, consider the second derivative of the function u with respect to t,

$$u_{tt}(x_i, t_j) = \frac{u(x_i, t_{j+1},) + u(x_i, t_{j-1}) - 2u(x_i, t_j)}{\Delta t^2}.$$

As we did for the space step, we can also rearrange this equation to solve for the next time step,

$$u(x_i, t_{j+1}) = u_{tt}(x_i, t_j)\Delta t^2 - u(x_i, t_{j-1}) + 2u(x_i, t_j).$$

Definitions, equations, and theorems listed in this chapter will be used throughout the rest of the thesis. The reader is encouraged to understand these preliminaries well and reference them throughout the thesis. These preliminaries will be used to derive a numerical scheme for a 2-D wave equation that will be implemented into Matrix Laboratories (MatLab).

CHAPTER 3

ANALYTICAL SOLUTIONS

In this chapter we will derive the wave equation and it's analytical solutions to the one and two dimensional wave equations. These derivations will further our understanding of wave equations in 1-D and 2-D which give us insight for our numerical solutions. These are the desired results that the numerical solution should approximate.

3.1 Derivation of the 1-D wave equation

In this section we will derive the 1-D wave equation, surprisingly only intermediate math and basic Physics are needed.

Let's remember that the the 1-D wave equation is commonly applied to describe phenomenons such as the waves in a musical guitar string. Therefore, let's consider a perfectly flexible elastic string stretched tightly between supports at the same horizontal level. We shall call the left endpoint x_0 and the right endpoint L. Let's also make a couple of assumptions that once the string is "plucked" that it will vibrate freely in a vertical plane provided dampening affects, such as air resistance, are neglected. We will also neglect the vertical forces from weight of the string. Let's consider an infinitesimally small piece of the string of length Δx and consider the forces that act on the points x and $x + \Delta x$. Let θ be the angle that is created from the tangent tension vector at the point (x, t) and a horizontal line at the point (x, t). Let $\theta + \Delta\theta$ be the angle created from the tension vector at point $(x + \Delta x, t)$ and the horizontal line a the point $(x + \Delta x, t)$. Also, let $u(x, t)$ be the vertical displacement of the point x at time t. These measurements can be seen in figure 3.1. This is an important idea to understand that each point of the string moves along a vertical line and not horizontally. We will let the tension in the string, which always acts in the tangential direction, be denoted by $T(x, t)$. Finally, ρ will represent the mass per unit length of the string.

Using Newton's law we can derive our first equation by setting the tension at each end our Δx piece of string equal to the product of the mass and the

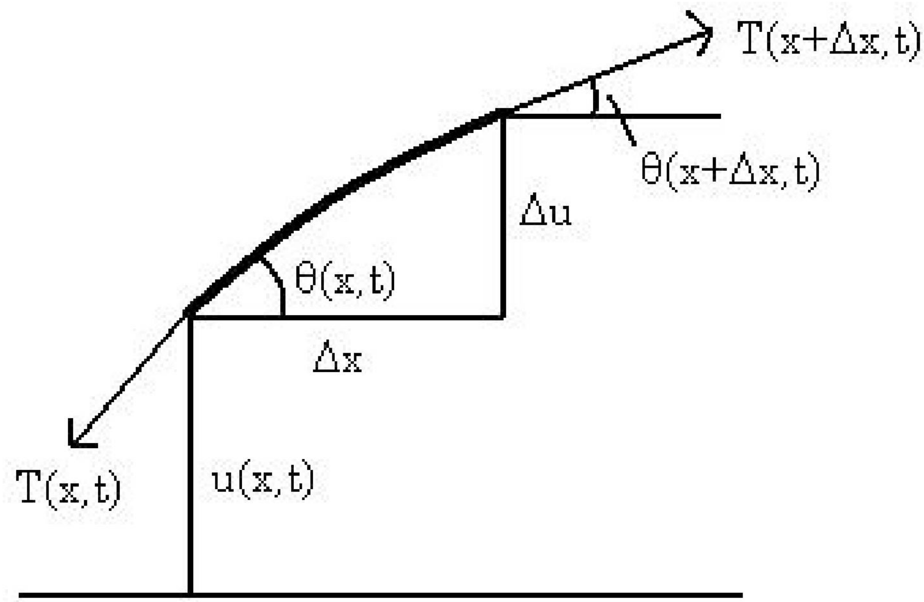

Figure 3.1: Piece of infinitesimal piece of string upon the pluck. Figure from Colin Mitchell.

acceleration of its mass center.

Therefore the horizontal components, which we will label as $H(t)$ will satisfy

$$H(t) = T(x+\Delta x, t)\cos(\theta+\Delta\theta) - T(x, t)\cos(\theta) = 0. \tag{3.1.1}$$

Again, this is because each point moves solely in the vertical direction.

Now, lets consider the vertical components of tension and apply Newton's laws, we obtain

$$T(x+\Delta x, t)\sin(\theta+\Delta\theta) - T(x, t)\sin(\theta) = \rho\Delta x u_{tt}(\bar{x}, t). \tag{3.1.2}$$

Where $\bar{x}$ denotes the center of gravity of the string we considered. Considering the vertical components of tension which we will now denote by V. We can write equation (3.1.2) as

$$\frac{V(x+\Delta x, t) - V(x,t)}{\Delta x} = \rho u_{tt}(\bar{x}, t).$$

The reader may notice that this was a convenient way for us to write this because if we take the limit we will obtain a first derivative, or in other words a velocity. Thus, taking the limit we obtain

$$V_x(x,t) = \rho u_{tt}(x,t). \tag{3.1.3}$$

We can conveniently write it in another form, that is

$$V(x,t) = H(t)\frac{\sin(\theta)}{\cos(\theta)} = H(t)u_x(x,t)$$

where H represents the horizontal component of tension and $\frac{\sin(\theta)}{\cos(\theta)}$ represents the slope, or derivative.

Now let's plug back into equation (3.1.3) and we obtain

$$(Hu_x)_x = \rho u_{tt}, \tag{3.1.4}$$

and since H is independent of x we obtain

$$Hu_{xx} = \rho u_{tt}. \tag{3.1.5}$$

The reader should notice that the wave equation is starting to look like its common form. Our final steps involve dividing by ρ and calling $\frac{H}{\rho} = c^2$. Thus we arrive at

$$u_{tt} = c^2 u_{xx}. \tag{3.1.6}$$

A final remark about c^2 is that H has tension units while ρ has mass/length which means c is a unit of velocity.

3.2 1-D Analytic solutions

3.2.1 d'Alembert solution

Take the 1-D wave equation $u_{tt} - c^2 u_{xx} = 0$ This factors nicely to

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

With a general solution of

$$u(x,t) = f(x+ct) + g(x-ct). \tag{3.2.1}$$

such that f, g are two arbitrary, twice differentiable, single valued functions. The function f is of a left traveling wave and g is the function of a right traveling wave. Figure 3.2 below is a simulation of a 1-D wave in which the reader can easily see f and g.

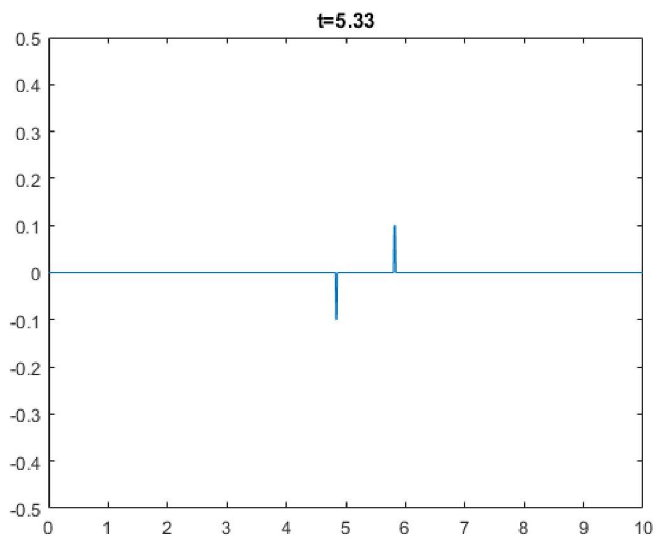

Figure 3.2: A model of the 1-D wave equation showing two waves f and g.

We are able to solve for f and g. In order to do so we will take our definition of 2.3.1, the 1-d wave equation and impose the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

By letting $t = 0$ in $u(x,t) = f(x+ct) + g(x-ct)$ we get $\phi(x) = f(x) + g(x)$. Taking the derivative of $u(x,t) = f(x+ct) + g(x-ct)$ we get

$$\psi(x) = cf'(x) - cg'(x)$$

Now let $\phi' = f' + g'$ and $\frac{1}{c}\psi = f' - g'$.

By adding and subtracting we obtain

$$f' = \frac{1}{2}(\phi' + \frac{\psi}{c}) \quad \text{and} \quad g' = \frac{1}{2}(\phi' - \frac{\psi}{c}).$$

and . Lets now introduce a variable s so we can integrate and obtain

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A$$

and

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B.$$

Note that A, B are constants and because $\phi(x) = f(x)+g(x)$ then $A+B = 0$ Now by replacing our dummy variable for f as $s = x + ct$ and for g as $s = x - ct$, we can substitute back into equation (3.2.1) and upon simplifying we obtain

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

This solution is also know as the d'Alembert solution.

Example 3.2.1 *Now that we have found an analytical solution to the 1-D wave equation we will do a quick example and verify that is satisfies the wave equation.*

Consider

$$u_{tt} - c^2 u_{xx} = 0 \quad for - \infty < x < \infty$$

and

$$u(x,0) = \phi(x) = 0, \quad u_t(x,0) = \psi(x) = \sin(x)$$

Thus substituting into the d'Alembert equation we obtain

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\,ds \tag{3.2.2}$$

$$= \frac{1}{2c}\left(-\cos(x+ct)+\cos(x-ct)\right). \tag{3.2.3}$$

After simplifying we obtain

$$u(x,t) = \frac{1}{c}\sin(x)\sin(ct)$$

Let's check our answer, observe $u_{xx} = \frac{-\sin(ct)\sin(x)}{c}$ *and* $u_{tt} = -c\sin(ct)\sin(x)$

Thus,

$$u_{xx}(x,t) = c^2\frac{-\sin(ct)\sin(x)}{c} = -c\sin(ct)\sin(x). \tag{3.2.4}$$

Thus $u(x,t)$ *satisfies the wave equation.*

Also, $u(x,0) = 0 = \phi(x)$ *and* $u_t(x,0) = \cos(ct)\sin(x) = \psi(x) = \sin(x)$ *as desired.*

Thus we conclude our example and say that $u(x,t) = \frac{1}{c}\sin(x)\sin(ct)$ satisfies the wave equation and its initial values.

3.2.2 Laplace transform

A basic understanding of the Laplace transform is assumed. An alternative to the d'Alembert solution is using the Laplace transform defined as

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s). \tag{3.2.5}$$

Where

$$\mathscr{L}\{u'(t)\} = sU(s) - u(0) \qquad \text{and} \qquad \mathscr{L}\{u''(t)\} = s^2U(s) - su(0) - u'(0). \tag{3.2.6}$$

Which follow immediately from integration by parts and assuming that the function is of exponential type. Pay special attention to the fact that a capital letter means it has already been through the transformation. We can apply this transform to the 1-D wave equation, but first, let's consider some initial conditions. Let

$$u(0,t) = f(t) \qquad \lim_{x\to\infty} u(x,t) = u(x,0) = \frac{\partial u}{\partial t} = 0.$$

Upon considering the 1-D wave equation $u_{tt} = c^2 u_{xx}$, we can apply the Laplace transform and obtain $\mathscr{L}\{u_{tt}\} = c^2\mathscr{L}\{u_{xx}\}$. Which, by equations (3.2.6) can be seen as

$$s^2\mathscr{L}\{u\} - su(x,0) - \frac{\partial u}{\partial t} = c^2\mathscr{L}\{u_{xx}\}$$

We notice that our initial conditions help simplify the above equation and we obtain

$$s^2\mathscr{L}\{u\} = c^2\mathscr{L}\{u_{xx}\}.$$

Let's now compute the Laplace transform of $\mathscr{L}\{u_{xx}\}$. Notice that

$$\mathscr{L}\{u_{xx}\} = \int_0^\infty e^{-st}\frac{\partial^2 u}{\partial^2 x}dt = \frac{\partial^2}{\partial^2 x}\int_0^\infty e^{-st}u(x,t)dt = \frac{\partial^2}{\partial^2 x}U(x,t)$$

Now setting the Left hand side equal to the right hand side we obtain

$$s^2 U = c^2\frac{\partial^2 U}{\partial^2 x}$$

This is an extremely beautiful step as we notice that this is an ordinary differential equation with a general solution of

$$U(x,s) = A(s)e^{\frac{sx}{c}} + B(s)e^{\frac{-sx}{c}}.$$

Now let's take the limit and apply our boundary conditions.

$$\begin{aligned}\lim_{x\to\infty} U(x,s) &= \lim_{x\to\infty}\int_0^\infty e^{-st}u(x,t)dt \\ &= \int_0^\infty \lim_{x\to\infty} u(x,t)dt \\ &= 0\end{aligned}$$

Thus $A(s) = 0$, $B(s) = F(s)$ and we arrive at

$$U(x,s) = F(s)e^{\frac{-xs}{c}}.$$

Now apply the inverse Laplace transform and second shifting theorem we arrive to

$$u(x,t) = \mathscr{H}(t - \frac{x}{c}) \sin(t - \frac{x}{c}) \tag{3.2.7}$$

Where $\mathscr{H}$ represents the Heavistep function defined as

$$\mathscr{H}(n) = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0. \end{cases}$$

3.3 Derivation of the 2-D wave equation

Applying the same ideas from our derivation of the 1-D wave equation we will now begin the derivation of the 2-D wave equation. We will consider a two dimensional membrane and impose a boundary condition on it. The reader is encourage to think of the membrane as the top of a drum. In this derivation our assumptions will be the following.

1. Density and tension is uniform and constant.
2. Membrane is perfectly flexible and fixed along boundary.
3. Out of plane deflections are small.

Our plane is in the x, y directions. As we did in section 3.1 lets begin considering an infinitesimally small section of this membrane. Instead of a length of string we will consider an area of a membrane, with lengths Δx and Δy. Thus the area will be $dA = (x, x+\Delta x) \times (y, y+\Delta y)$. Let's call $z = u(x, y)$, displacement from rest, and lets now begin considering all the forces acting upon the membrane.

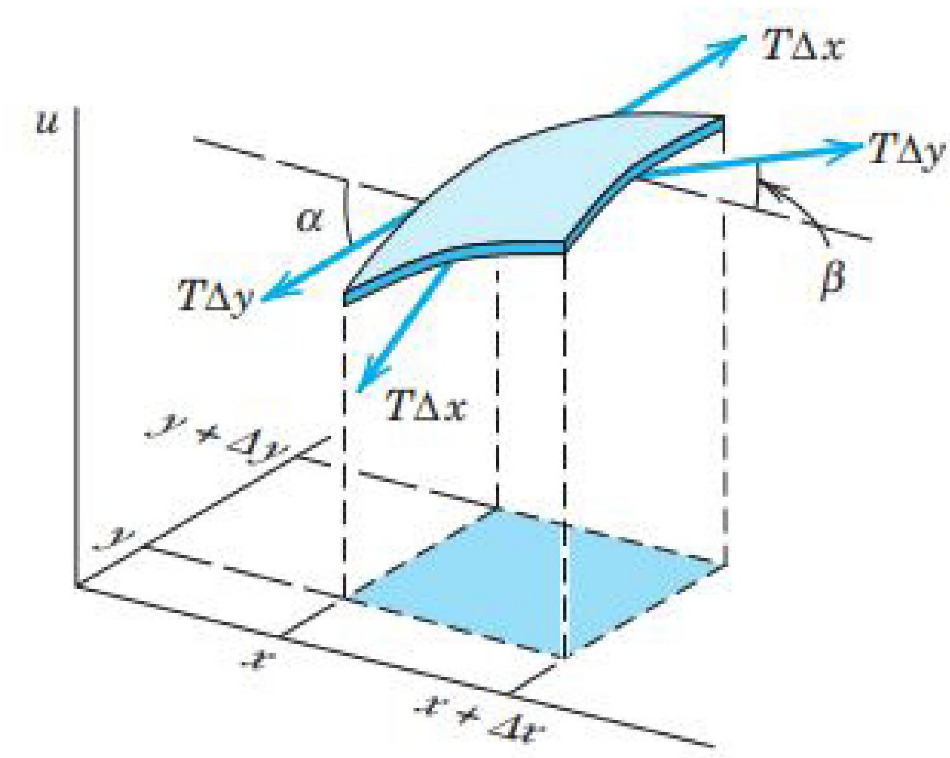

Figure 3.3: 3 dimensional sketch of a membrane being stretched. Figure obtained from [11].

As we noticed in the 1-D derivation each point of the membrane will not move horizontally, each point will move only in the vertically directions. We will take a closer look at figure 3.3 by observing it in the $x - z$ plane.

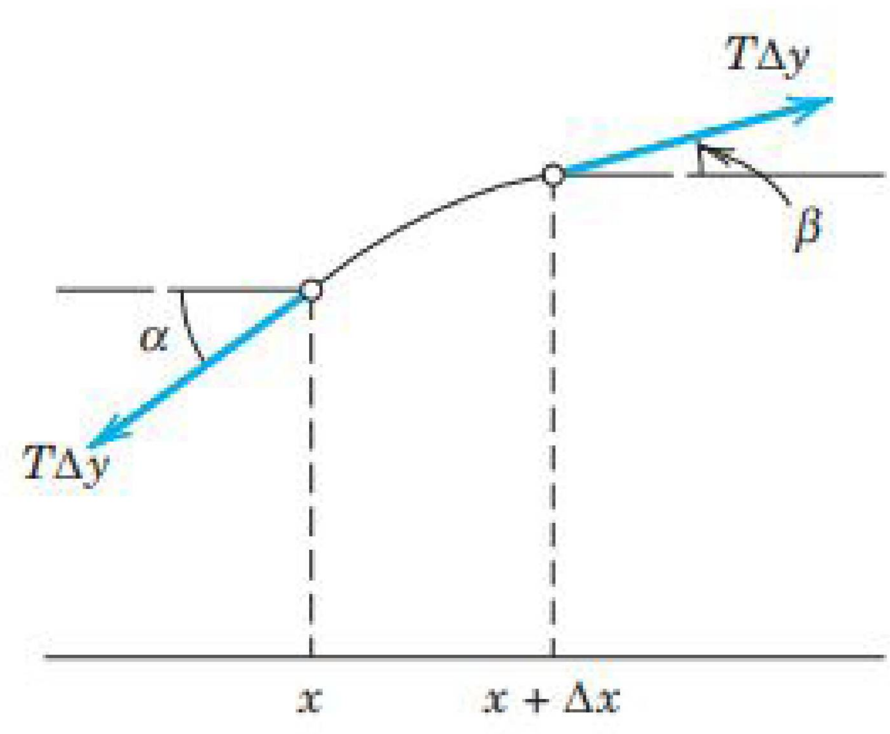

Figure 3.4: Figure 3.3 seen in the $x-z$ plane.Figure obtained from [11]

Thus, by small angle approximations the sum of the forces in the x-direction can be written as

$$\sum_x F = T\Delta y\cos(\beta) - T\Delta y\cos(\alpha) = 0.$$

For the vertical forces we will consider the sum of forces acting on the left and right sides of the membrane, then the front and back sides of the membrane. Lets begin with the left and right sides,

$$\sum_{y-L\&R} F = T\Delta y\sin(\beta) - T\Delta y\sin(\alpha). \tag{3.3.1}$$

By small angle approximations which state $\sin(\theta) \approx \theta \approx \tan(\theta)$ we obtain

$$\sum_{y-L\&R} = T\Delta y[\tan(\beta) - \tan(\alpha)]. \tag{3.3.2}$$

Observe that

$$\tan(\beta) = u_x \Big|_{x=x+\Delta x}^{y=y_1 \in [y, y+\Delta y]} = u_x(x+\Delta x, y_1) \tag{3.3.3}$$

and

$$\tan(\alpha) = u_x \Big|_{x=x}^{y=y_2 \in [y, y+\Delta y]} = u_x(x, y_2). \tag{3.3.4}$$

Notice that equations (3.3.3)and (3.3.4) state that the derivative or velocity is dependent on some value of y. Just as we did in the 1-D case lets plug back into equation (3.3.2),

$$T\Delta y[u(x+\Delta x, y_1) - u(x, y_2)]. \tag{3.3.5}$$

Using the same method lets consider the forces on the front and back of the membrane acting in the y direction,

$$\sum_{y-F\&B} F = T\Delta x[u_y(x_1, y+\Delta y) - u_y(x_2, y)]. \tag{3.3.6}$$

Lets now apply Newton's law to the vertical forces,

$$T\Delta y[u(x+\Delta x, y_1) - u(x, y_2)] + T\Delta x[u_y(x_1, y+\Delta y) - u_y(x_2, y)] = \rho \Delta x \Delta y u_{tt}. \tag{3.3.7}$$

Lets divide each side by $\Delta x \Delta y \rho$ to obtain

$$\frac{T}{\rho}\left[\frac{u_x(x+\Delta x, y_1) - u(x, y_2)}{\Delta x} + \frac{u_y(x_1, y+\Delta y) - u_y(x_2, y)}{\Delta y}\right] = u_{tt} \tag{3.3.8}$$

We hope the reader becomes especially giddy as they notice the elegance of equation (3.3.8). As the limit of Δx and Δy go to zero we obtain

$$\frac{T}{\rho}\left(u_{xx} + u_{yy}\right) = u_{tt}. \tag{3.3.9}$$

Finally let $c^2 = \frac{T}{\rho}$ and we arrive at the 2-D wave equation we are familiar with

$$c^2\left(u_{xx} + u_{yy}\right) = u_{tt}. \tag{3.3.10}$$

3.4 2-D Analytic solution

Consider the 2-D wave equation

$$u_{tt} - c^2(u_{xx} + u_{yy}) = 0.$$

With $u_{tt} = c^2 \Delta u$, $u(x, y, 0) = u_t(x, y, 0) = u(0, y, t) = u(l, y, t) = 0$ on the boundary of $[0, a] \times [0, b]$. With a domain of $(x, y) \in [0, a] \times [0, b]$ and $t > 0$.

Let $u(x, y, t) = X(x)Y(y)T(t)$. Then we can replace $u_{tt} - c^2(u_{xx} + u_{yy}) = 0$ with

$$XYT'' = c^2(X''YT + XY''T)$$

Divide 3.4 both sides by $c^2 u(x, y, t)$ $$\frac{T''}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y}$$

Set equation equal to A $$\frac{T''}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = A$$

Let $B = \frac{X''}{X}$ and $C = \frac{Y''}{Y}$ $$X'' - Bx = 0 \text{ and } Y'' - CY = 0$$

Thus $$C = A - B$$

From the boundary conditions $X(0) = X(a) = Y(0) = Y(b) = 0$.

Applying these conditions we find infinitely many nontrivial solutions indexed by m and n. Thus, solving for $X(x)$ and $Y(y)$ we obtain

$$X_m(x) = \sin(\mu_m(x)), \qquad \mu_m = \frac{m\pi}{a}, \qquad m = 1, 2, 3, \ldots$$

$$Y_n(y) = \sin(\upsilon_n y), \qquad \upsilon_n = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \ldots$$

$$B = -\mu_m^2, \qquad C = -\upsilon_n^2.$$

Now that we have solved for $X(x)$, $Y(y)$, B and C, lets do the same for $T(t)$.

Rearrange and simplify (3.4) to obtain

$$T'' - c^2 AT = 0.$$

Remember that $C = A - B$ or $A = B + C = -(\mu_m^2 + \upsilon_n^2)$. With this

$$T_{mn}(t) = B_{mn} cos\lambda_{mn} t + B*_{mn} sin\lambda_{mn} t$$

and

$$\lambda_{mn} = c(\mu_m^2 + \upsilon_n^2)^{\frac{1}{2}} = c\pi(\frac{m^2}{a^2} + \frac{n^2}{b^2}).$$

All together we obtain,

$$
\begin{aligned}
u_{mn}(x,y,t) &= X_m(x)Y_n(y)T_{mn}(t) \\
&= \sin(\mu_m(x))\sin(\upsilon_n y)\left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right).
\end{aligned}
$$

Thus the general solution becomes

$$
u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin(\mu_m(x))\sin(\upsilon_n(y))\left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right). \tag{3.4.1}
$$

In order to find B and B^*we apply initial conditions to f and g.

$$
f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn} sin\frac{m\pi}{a} x sin\frac{n\pi}{b} y
$$

$$
g(x,y) = u_t(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{mn} B^*_{mn} sin\frac{m\pi}{a} x sin\frac{n\pi}{b} y
$$

By orthogonality, the functions

$$
Z_{mn}(x,y) = \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y) \quad m,n \in \mathbb{N},
$$

are pairwise orthogonal relative to the inner product

$$
\langle f,g\rangle = \int_0^a \int_0^b f(x,y)g(x,y)dydx.
$$

By this theorem the Fourier coefficient becomes

$$\begin{aligned}
B_{mn} &= \frac{\langle f, Z_{mn} \rangle}{Z_{mn}, Z_{mn}} \\
&= \frac{\int_0^a \int_0^b f(x,y)\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\,dydx}{\int_0^a \int_0^b \sin^2(\frac{m\pi}{a}x)\sin^2(\frac{n\pi}{b}y)\,dydx} \\
&= \frac{4}{ab}\int_0^a \int_0^b f(x,y)\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\,dydx
\end{aligned}$$

and

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^a \int_0^b g(x,y)\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\,dydx.$$

In conclusion, the analytical solution to the 2-D wave equation is given by (3.4.1) with $B_{mn}, B^*_{mn}, \mu_m, \nu_n, \lambda_{mn}$ as defined above.

Example 3.4.1 *Consider a rectangular membrane* 3×3 *with* $c = 9$*. We will deform the membrane to fit* $f(x,y) = xy(3-y)(3-y)$*,* $g(x,y) = 0$*. Keep edges fixed, then release at* $t = 0$*.*

Notice that since $g(x,y) = 0$*, there is no initial velocity. Also, we get* $B^*_{mn} = 0$*. We obtain* B_{mn} *by the following,*

$$\begin{aligned}
B_{mn} &= \frac{4}{3*3}\int_0^3 \int_0^3 xy(3-x)(3-y)\sin(\frac{m\pi}{3}x)\sin(\frac{n\pi}{3}y)\,dydx \\
&= \frac{4}{9}\left[\int_0^3 \int_0^3 x(3-x)\sin(\frac{m\pi}{3}x)\,dx \int_0^3 y(3-y)\sin(\frac{n\pi}{3}y)\,dy\right] \\
&= \frac{4}{9}\left[\frac{54(1+(-1)^{m+1}}{\pi^3 m^3}\frac{54(1+(-1)^{n+1}}{\pi^3 n^3}\right] \\
&= \frac{1296}{\pi^6}\frac{(1+(-1)^{m+1})(1+(-1)^{n+1})}{m^3 n^3}.
\end{aligned}$$

Next we will compute λ_{mn},

$$\lambda_{mn} = 9\pi\left(\frac{m^2}{9} + \frac{n^2}{9}\right)^{\frac{1}{2}} = 3\pi\sqrt{m^2 + n^2}.$$

All together we obtain

$$u(x, y, t)$$
$$= \frac{1296}{\pi^6}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(1+(-1)^{m+1})(1+(-1)^{n+1})}{m^3n^3}\sin(\frac{m\pi}{3}x)\sin(\frac{n\pi}{3}y)\cos(3\pi\sqrt{m^2+n^2}\,t)\Big).$$

Thus we have come up with a solution for any $t > 0$.

3.5 Derivation of the 2-D Wave equation in polar coordinates

In this section of the thesis we will convert the Laplacian of the wave equation to polar coordinates. Although this is not necessary to model the 2-D wave equation numerically it could provide insight to future problems to be solved perhaps in a Phd. As the reader could imagine polar coordinates would be beneficial if the region was in the shape of a disk.

Consider the Laplacian as

$$\Delta(u) = u_{xx} + u_{yy} = 0 \tag{3.5.1}$$

We know that we can put x and y coordinates in terms of θ as the following.

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \tag{3.5.2}$$

Now, let's compute some important partial derivatives

$$\frac{\partial x}{\partial r} = \cos(\theta) \qquad \frac{\partial x}{\partial \theta} = -r\sin(\theta) \tag{3.5.3}$$

$$\frac{\partial y}{\partial r} = \sin(\theta) \qquad \frac{\partial y}{\partial \theta} = r\cos(\theta) \tag{3.5.4}$$

Our goal is to obtain $\frac{\partial^2 u}{\partial r^2}$ and $\frac{\partial^2 u}{\partial \theta^2}$.

Let's start with $\frac{\partial u}{\partial r}$. By the chain rule,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \tag{3.5.5}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \tag{3.5.6}$$

$$= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}. \tag{3.5.7}$$

Continuing on to $\frac{\partial^2 u}{\partial r^2}$,

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial y} \tag{3.5.8}$$

$$= \cos\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}\right) \tag{3.5.9}$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}. \tag{3.5.10}$$

We are about halfway done, now we should focus on $\frac{\partial^2 u}{\partial\theta^2}$. We will use the same approach that we took to solve for $\frac{\partial^2 u}{\partial r^2}$.

$$\frac{\partial u}{\partial\theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta} \tag{3.5.11}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta \tag{3.5.12}$$

$$= -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}. \tag{3.5.13}$$

Continuing on to $\frac{\partial^2 u}{\partial\theta^2}$,

$$\begin{aligned}
\frac{\partial^2}{\partial\theta^2} &= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial}{\partial\theta}\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\frac{\partial}{\partial\theta}\frac{\partial u}{\partial y}\\
&= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial\theta}\right) - r\sin\theta\frac{\partial u}{\partial y}\\
&+ r\cos\theta\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial\theta}\right)\\
&= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2}{\partial x\partial y}r\cos\theta\right)\\
&- r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\left(\frac{\partial^2 u}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}r\cos\theta\right)\\
&= -r\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right) + r^2\left(\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}\right).
\end{aligned}$$

Let's now divide both sides by r^2 and apply equation 3.5.7 to obtain

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + cos^2\theta\frac{\partial^2 u}{\partial y^2}. \tag{3.5.14}$$

Now lets add (3.5.10), (3.5.14) and apply one of the most beautiful trigonometric identities, $cos^2\theta + \sin^2\theta = 1$. By doing so we will obtain,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{-1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Rearranging and using different notation we obtain the Laplcian

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{3.5.15}$$

Now let's apply (3.5.15) to the wave equation to obtain

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right). \tag{3.5.16}$$

Thus equation (3.5.16) is the wave equation written in polar coordinates. We will now find solutions in the following section by separation of variables.

3.6 2-D Polar coordinate solution

First, let's assume homogeneous boundary conditions, that is $u(\alpha, \theta, t) = 0$. Where, a is the edge of the disk. Then,

$$u(r,\theta,0) = f(r,\theta) \quad \text{and} \quad u_t(r,\theta,0) = g(r,\theta).$$

Separating with respect to time we obtain

$$u(r,\theta,t,) = R(r)\Theta(\theta)T(t) = \phi(r,\theta)T(t).$$

To work with equation in a simpler form let's write equation (3.5.16) as the following

$$u_{tt} = c^2\left(\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}\right).$$

Upon apply our seperation of variables we obtain

$$\phi T'' = c^2\left(\frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta}\right)T.$$

Continuing,

$$\frac{T''}{c^2T} = \frac{\frac{1}{r}r\phi_r)_r + \frac{1}{r^2}\theta\theta}{\phi} = -\lambda.$$

Upon dividing we obtain our first separated equation for T,

$$T'' + \lambda c^2 T = 0. \tag{3.6.1}$$

Notice that if we multiply equation (3.6.1) by $r^2\phi$ we will obtain the Helmholtz equation which is written as,

$$r^2\phi_r r + r\phi_r + \phi_{\theta\theta} + \lambda r^2\phi = r^2R''\Theta + rR - \Theta + R\Theta^{''} + \lambda r^2 R\Theta = 0. \tag{3.6.2}$$

We will now begin the separation process for Θ by dividing equation (3.6.2) by $R\Theta$, upon rearraniging we obtain

$$\frac{r^2R^{''} + rR' + \lambda r^2 R}{R} = \frac{-\Theta^{''}}{\Theta} = \mu.$$

Therefore we can solve for Θ as

$$\Theta^{''} + \mu\Theta = 0. \tag{3.6.3}$$

There are a couple of things that we know about Θ.

$$\Theta = a_n \cos n\theta + b_n \sin\theta \quad \text{and} \quad \mu = 0^2, 1^2, 2^2, ..., n^2, ..$$

We will utilize $\mu = n^2$ to help solve for R. Doing so we obtain

$$r^2 R'' + rR' + (\lambda r^2 - n^2)R = 0. \tag{3.6.4}$$

Since we set the edge of our circle to be zero, let's restate that $R(a) = 0$ and $R(0)$ is bounded. Let's also get ride of λ by setting ☺ $= \sqrt{\lambda r}$. By doing so we obtain

$$\frac{dR}{dr} = \frac{dR}{d☺}\frac{d☺}{dr} = \sqrt{\lambda}\frac{dR}{d☺} \quad \text{and} \quad \frac{d^2R}{dr^2} = \lambda\frac{d^2R}{x☺^2}.$$

Upon substitution we obtain

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda r^2 - n^2)R = ☺^2\frac{d^2R}{d☺^2} + ☺\frac{dR}{d☺} + (☺^2 - n^2)R = 0, \tag{3.6.5}$$

as desired. We observe that equation (3.6) is known as Bessel's equation of order n. Derivations of the solutions to the Bessel's equation will not be shown here as our focus is to derive the solution for polar coordinates. Derivation of the solutions to Bessel's equation would need a section of its own, however we will mention that the solutions are derived from recurrence relations and state it has the solution $cJ_n(x) = cJ_n(\sqrt{\lambda r})$.

We are getting close to our solution, let's recall boundary conditions for $J_n(\sqrt{\lambda}a) = 0$ to obtain

$$\lambda_{m,n} = \left(\frac{z_{m,n}}{a}\right)^2 \quad \text{for} \quad n \geq 0 \quad \text{and} \quad m \geq 1.$$

Which has eigenfunctions of

$$R_{m,n}(r) = J_n\left(\frac{z_{m,n}}{a}r\right).$$

Now that we have λ and our eigenfunctions we can finish solving equation (3.6.1) and obtain

$$T_{m,n}(t) = A\cos\left(\sqrt{\lambda_{m,n}}ct\right) + B\sin\left(\sqrt{\lambda_{m,n}}ct\right).$$

Thus our solution becomes

$$\begin{aligned}
u(r,\theta,t) &= f(r,\theta) + g(r,\theta) \\
&= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\sqrt{\lambda_{m,n}}r\right)\cos\left(\sqrt{\lambda_{m,n}}ct\right)\left(a_{m,n}\cos n\theta + b_{m,n}\sin n\theta\right) \\
&+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\sqrt{\lambda_{m,n}}r\right)\sin\left(\sqrt{\lambda_{m,n}}ct\right)\left(c_{m,n}\cos n\theta + d_{m,n}\sin n\theta\right).
\end{aligned}$$

Where the coefficients $a_{m,n}$, $b_{m,n}$, $c_{m,n}$, $d_{m,n}$ are determined from Fourier series. The process of finding the coefficients is not difficult, however it can be quite exhausting which is why we will leave it to the reader to compute them if desired.

Now that we have derived a couple different types of analytical solutions to the 1 and 2 dimensional wave equations we will continue to chapter four where we will work on a numerical method that will approximate solutions to the analytical wave equation solutions. Our numerical methods will focus on 1 and 2 dimensional waves in the Cartesian plane and a finite difference method will be used to approximate solutions.

CHAPTER 4

DERIVING NUMERICAL SCHEMES

In this section we will introduce a new way to obtain solutions to the wave equation using a derivation of Taylor's theorem 2.1.1.

As stated before, we will be using the finite difference method to discretize our numerical solutions. Although other methods could be of interest we will proceed with this method. Another important thing the reader should consider is that since we are working with computers we are restricted to a finite discrete interval. Because of this, we are forced to come up with a scheme that approximates the actual solution. In a perfect world, and likely very distant future these numerical schemes will be unnecessary and we will be able to produce numerically the analytical solutions.

Finally, the reader may notice that the "big oh" has been dropped in these approximations.

4.1 1-D Numerical derivation

As a gentle reminder the 1-D wave equation is of the form $u_{tt} = c^2 u_{xx}$.

Lets begin by introducing $u(x,t)$, a function that has a spatial variable x and a time variable t, and suppose u is continuous with respect to both x and t. Now consider $u(x_i, t_j)$, still the same function but this time x_n and t_m are discrete values which are determined by the grid lines. i.e. $x_0 = a < x_1 < x_2 < ... < x_{i-1}, x_i < x_{i+1} ... < x_n = b$. The index x_i is used to represent the function at the current spatial step, while x_{i-1} and x_{i+1} will be used to represent the previous and following spatial step. The same method is also applied to the time variable t.

Applying the manuever of 2.3.11 to $u(x_i, t_j)$, it can easily be seen that $u(x_i, t_j)$ has the following second partial derivatives.

For a fixed j,

$$u_{xx}(x_i, t_j) = \frac{u(x_{i+1}, t_j) + u(x_{i-1}, t_j) - 2u(x_i, t_j)}{\Delta x^2}.$$

For a fixed i,

$$u_{tt}(x_i, t_j) = \frac{u(x_i, t_{j+1}) + u(x_i, t_{j-1}) - 2u(x_i, t_j)}{\Delta t^2}.$$

Thus subsisting into $u_{tt} = c^2 u_{xx}$ we can rewrite the 1-D wave equation as

$$\frac{u(x_i, t_{j+1}) + u(x_i, t_{j-1}) - 2u(x_i, t_j)}{\Delta t^2} = c^2 \left(\frac{u(x_{i+1}, t_j) + u(x_{i-1}, t_j) - 2u(x_i, t_j)}{\Delta x^2} \right).$$

Applying 2.3.12 to the wave equation we can solve for the next time step by subtracting $\frac{u(x_i, t_{j-1}) - 2u(x_i, t_j)}{\Delta t^2}$ from each side of the equation above,

$$\begin{aligned} &\frac{u(x_i, t_{j+1}) + u(x_i, t_{j-1}) - 2u(x_i, t_j)}{\Delta t^2} - \frac{u(x_i, t_{j-1}) - 2u(x_i, t_j)}{\Delta t^2} \\ &= c^2 \left(\frac{u(x_{i+1}, t_j) + u(x_{i-1}, t_j) - 2u(x_i, t_j)}{\Delta x^2} \right) \\ &\quad - \frac{u(x_i, t_{j-1}) - 2u(x_i, t_j)}{\Delta t^2}. \end{aligned}$$

Solving for the next time step,

$$\begin{aligned} u(x_i, t_{j+1}) = c^2 &\left(\frac{u(x_{i+1}, t_j) + u(x_{i-1}, t_j) - 2u(x_i, t_j)}{\Delta x^2} \right) \Delta t^2 \\ &- \frac{u(x_i, t_{j-1} - 2u(x_i, t_j)}{\Delta t^2} \Delta t^2. \end{aligned}$$

Finally, canceling out Δt^2 we obtain

$$\boxed{u(x_i, t_{j+1}) = c^2 \left(\frac{u(x_{i+1}, t_j) + u(x_{i-1}, t_j) - 2u(x_i, t_j)}{\Delta x^2} \right) \Delta t^2 - u(x_i, t_{j-1}) + 2u(x_i, t_j).} \tag{4.1.1}$$

This is the exact equation that we will use in our code. Notice that above in equation (4.1.1) we have solved for the next time step and found that it it is equal to the function evaluated at its previous, present, and next space steps along with the constant c and grid width Δx and Δt. This process could have also been done to solve for the next space step, in fact the reader is encouraged to do so as an excercise.

4.2 1-D Numerical Scheme

Recall that $u(x_i, t_j)$ approximates the analytic solution $u(x, t)$ at the mesh/-grid point (x_i, t_j). The code presented below has been adopted from Haroon Stephens YouTube video https://www.youtube.com/watch?v=O6fqBxuM-g8. Notation in the code becomes `w(i,j)`. Below is the Matlab code for the 1-D wave equation, a similar approach will be used to help derive the 2-D wave equation. Equation (4.1.1) can be seen under the documentation in the preamble involved in the source section of the code as well as the simplifying assumption of $\Delta t = \Delta x$. A more detailed explanation of how the code works is saved for section 4.4.

```
%%Solving a PDE
clear;
%Equation wtt=c^2wxx+f
%%Domain
%Space
Lx=10;
dx=.01;
nx=fix(Lx/dx);
x=linspace(0, Lx, nx);
%Time
T=20;
%%Field Variable
%Variables
wn=zeros(nx,1);
wnm1=wn; % w at time n-1
wnp1=wn; % w at time n+1
%Parameters
CFL=1; %CFL=c.dt/dx
c=1;
dt=CFL*dx/c;


%% Initial Conditions
wn(49:51)=[0.1 0.2 0.1];
wnp1(:)=wn(:);
%%Time stepping Loop
t=0;
while(t < T)
```

```
    %Reflecting Boundary Conditions
    wn([1 end])=0;
    %solution
    t=t+dt;
    wnm1=wn; wn=wnp1; %Save current and previous arrays
    %Source
    %wn(50)=dt^2*20*sin(20*pi(t/T);
    for i=2:nx-1
        wnp1(i) = 2*wn(i)-wnm1(i) ...
            +CFL^2 * (wn(i+1) - 2*wn(i) + wn(i-1));
    end
    %Visualize at selected steps
    clf;
    plot(x,wn);
    title(sprintf('t=%.2f',t));
    axis([0 Lx -0.5 0.5]);
    shg; pause(0.01);
end
```

4.3 2-D Numerical derivation

As a gentle reminder the 2-D wave equation is of the form $u_{tt} = c^2(u_{xx} + u_{yy})$.

We can derive numerical solutions to the 2-D wave equations similarly to the 1-D wave equations. It is important to mention that for the 2-D wave equation we have introduced a new spatial variable y. The indices of y will also behave similarly to the indices of x and t described in section 4.1.

For a fixed i and k,

$$u_{tt}(x_i, t_j, y_k) = \frac{u(x_i, t_{j+1}, y_k) + u(x_i, t_{j-1}, y_k) - 2u(x_i, t_j, y_k)}{\Delta t^2}.$$

For a fixed j and k,

$$u_{xx}(x_i, t_j, y_k) = \frac{u(x_{i+1}, t_j, y_k) + u(x_{i-1}, t_j, y_k) - 2u(x_i, t_j, y_k)}{\Delta x^2}.$$

For a fixed i and j,

$$u_{yy}(x_i, t_j, y_k) = \frac{u(x_i, t_j, y_{k+1}) + u(x_i, t_j, y_{k-1}) - 2u(x_i, t_j, y_k)}{\Delta y^2}.$$

Thus substituting into the 2-D wave equation we obtain

$$\begin{aligned}
&\frac{u(x_i, t_{j+1}, y_k) + u(x_i, t_{j-1}, y_k) - 2u(x_i, t_j, y_k)}{\Delta t^2} \\
&= c^2 \left(\frac{u(x_{i+1}, t_j, y_k) + u(x_{i-1}, t_j, y_k) - 2u(x_i, t_j, y_k)}{\Delta x^2} \right. \\
&\left. + \frac{u(x_i, t_j, y_{k+1}) + u(x_i, t_j, y_{k-1}) - 2u(x_i, t_j, y_k)}{\Delta y^2} \right).
\end{aligned}$$

Just like we did for the 1-D wave equation we can also solve for the next time step in the 2-D wave equation. We will begin the Algebraic process of solving for the next time step $u(x_i, t_{j+1}, y_k)$. By subtracting

$$\frac{u(x_i, t_{j-1}, y_k) - 2u(x_i, t_j, y_k)}{\Delta t^2}$$

from each side, we obtain

$$\begin{aligned}
&\frac{u(x_i, t_{j+1}, y_k) + u(x_i, t_{j-1}, y_k) - 2u(x_i, t_j, y_k)}{\Delta t^2} - \frac{u(x_i, t_{j-1}, y_k) - 2u(x_i, t_j, y_k)}{\Delta t^2} \\
&\quad = c^2 \left(\frac{u(x_{i+1}, t_j, y_k) + u(x_{i-1}, t_j, y_k) - 2u(x_i, t_j, y_k)}{\Delta x^2} \right. \\
&\qquad \left. + \frac{u(x_i, t_j, y_{k+1}) + u(x_i, t_j, y_{k-1}) - 2u(x_i, t_j, y_k)}{\Delta y^2} \right) \\
&\qquad - \frac{u(x_i, t_{j-1}, y_k) - 2u(x_i, t_j, y_k)}{\Delta t^2}.
\end{aligned}$$

We will finish solving by canceling the two terms on the left hand side of the equation, using the simplifying assumption $\Delta x = \Delta y$, and multiplying each side by Δt^2.

$$\begin{aligned}
u(x_i, t_{j+1}, y_k) = c^2 &\left(\frac{u(x_{i+1}, t_j, y_k) + u(x_{i-1}, t_j, y_k) - 2u(x_i, t_j, y_k)}{\Delta x^2} \right. \\
&\left. + \frac{u(x_i, t_j, y_{k+1}) + u(x_i, t_j, y_{k-1}) - 2u(x_i, t_j, y_k)}{\Delta y^2} \right) \Delta t^2 \\
&- \frac{u(x_i, t_{j-1}) - 2u(x_i, t_j)}{\Delta t^2} \Delta t^2.
\end{aligned}$$

Finally, by adding like terms,rearranging and canceling products we obtain

$$\boxed{\begin{aligned} u(x_i, t_{j+1}, y_k) = c^2 &\left(\frac{u(x_{i+1}, t_j, y_k) + (x_i, t_j, y_{k+1}) - 4u(x_i, t_j, y_k)}{\Delta x^2} \right) \Delta t^2 \\ &+ c^2 \left(\frac{u(x_{i-1}, t_j, y_k) + u(x_i, t_j, y_{k-1})}{\Delta x^2} \right) \Delta t^2 - u(x_i, t_{j-1}) + 2u(x_i, t_j). \end{aligned}}$$

Notice that this is the next step for time. This process could have been done to solve for the next time step for either x or y. We will use this theoretical derivation of the next time step in section 4.4 to implement the 2-D wave equation in Matlab.

4.4 2-D Numerical Scheme

Using the 1-D Matlab code and our acquired knowledge we are now ready to come up with the code for the 2-D wave equation. The reader is encouraged to reference the code listed below while it is explained in this section.

To begin, let's start with our domain to help establish grid and mesh points. Let a, b, c, d be endpoints of our region. That is, $a \leq x \leq b$ and $c \leq y \leq d$. We will assign `Lx` as $|b - a|$ and `Ly` as $|d - c|$. Now that we have a region, lets start to "slice it up" in order to make a grid. Let `dx` and `dy` represent our values of h and k respectively, as defined in definition 2.3.6. Since $dx = \frac{b-a}{n}$ we are able to solve for n and we achieve $n = \frac{b-a}{dx}$. In our code we will assign n as `nx=fix(Lx/dx)` . We will also do the same for m by assigning `ny=fix(Ly/dy)`. Also, let $m = n$. Thus, `Lx=Ly` and `dx=dy`. Finally, for the grid lines we will create a vector of points ranging from `0` to `nx` spaced by `Lx`. Thus we will assign `x=linspace(0,Lx,nx)`. Similarly we will do the same for y and achieve `y=linspace(0,Ly,ny)`.

In the time section of the code we will chose `T` as final time to be whatever length we desire.

In the field variable section of the code we will assign function values to a matrix which are crucial in the while loop of the code. The command `wn=zeros(nx,ny)` creates a matrix of zeros of width `nx` and length `ny`. These are the outputs of `w=w(i,j)` at the current time step. Also note that in this section the previous time step will be assigned as `wnm1=wn`, and the next time step will be assigned as `wnp1=wn`. This will have great importance to us in the while loop of the code.

In the parameters section of the code we define our Courant-Friedrichs-Lewy number as `CFL=c*dt/dx`. Notice here that even though earlier we made a simplifying assumption of `dx=dy` we did not do the same for `dt`. This is done so that the time step must be less than a certain time for stability. Next we will chose a value for `c`, remember `c` is our constant. Finally, we will define our time step at `CFL*dx/c`.

The code continues, declaring `t=0` as our initial time for the while loop. The first thing to notice in the while loop is that there are two options for boundary conditions in the code. The user can chose either reflecting or absorbing boundary conditions. For reflecting boundary conditions, since our figure has finite boundaries we will implement Dirichlet boundary conditions. This definition will help us specify the values that a solution needs to take on along the boundary of the domain, which is precisely what is done in the code i.e. `wn(:, [1 end])=0` and `wn([1 end],:)=0`. To give the reader a better understanding of what the computer is doing to the matrix `wn` refer to the figure below.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Imagine `wn` is the matrix on the left. It shows the values associated with the mesh points. That is, the columns represent x_i and the rows represent y_j thus `wn(j,i)` is the mesh point at `(j,i)`. The matrix on the right is what happens when the Dirichlet boundary conditions are imposed. As one can see it simply sets all of the values on the boundaries equal to zero.

In the next section of the code the step for `t` in the time loop is given as `t=t+dt` and after boundary conditions are applied previous and current values in the time step matrix are recorded as `wnm1=wn` and `wn=wnp1`.

Next, a source will be introduced. We are able to chose the location of the source thus this will assign values to the original `wn` matrix. That is, the initial condition or an external source. One interesting aspect is that we are able to assign the source as a constant, or we can assign it to a function. For example, if we assign `wn(50,50)=10` this will make the mesh point `(50,50)=10`. However, if we assign `wn(50,50)=sin(dt)`, this will make the point `(50,50)` a function of $sin(dt)$ and the sin wave will be used for the duration of `T` to propagate waves.

Perhaps the most beautiful part of the whole thesis is now implemented in the for loop. Notice that the for loop was designed to assign values to the next time step matrix. By this point of the thesis the reader should have no problem recognizing equation (4.3) in the for loop. The only difference is that instead of c in equation (4.3) the CFL is listed instead. This is alright because in the code the `CFL=.5` and `c=1`. This returns a correct value for the CFL. That is, $CFL = c\frac{\Delta t}{\Delta x}$. The reader is further encouraged to spend extra time admiring the beauty of this section of the code.

The rest of the code is computer mumbo jumbo to produce the results as figures. If the reader is interested in this section of the code they are referred to a matlab handbook.

Below is the code for the 2-D equation in Matlab.

```
%% Solving a PDE
clear;
% Equation
%% wtt = c^2 wxx + c^2 wyy +f
%% Prepare the new file.
 vidObj = VideoWriter('wave.avi');
 open(vidObj);
%% Domain
% Space
Lx=1;
Ly=1;
dx=0.01;
dy=dx;
nx=fix(Lx/dx);
ny=fix(Ly/dy);
x=linspace(0, Lx, nx);
y=linspace(0, Ly, ny);
```

```
% Time
T=100;
%% Field variable
% Variables
wn=zeros(nx,ny);
wnm1=wn; % w at time n-1
wnp1=wn; % w at time n+1
% Parameters
CFL=0.5; % CFL = c.dt/dx
c=1;
dt=CFL*dx/c;
%% Initial Conditions
%% Time Stepping Loop
t=0;
while(t < T)
    % Refelecting Boundary Conditions
    %wn(:,[1 end])=0;
    %wn([1 end],:)=0;
    % Absorbing boundary conditions
    wnp1(1,:)=wn(2,:) + ((CFL-1)/(CFL+1))*(wnp1(2,:)-wn(1,:));
    wnp1(end,:)=wn(end-1,:) + ((CFL-1)/(CFL+1))*(wnp1(end-1,:)-wn(end,:));
    wnp1(:,1)=wn(:,2) + ((CFL-1)/(CFL+1))*(wnp1(:,2)-wn(:,1));
    wnp1(:,end)=wn(:,end-1) + ((CFL-1)/(CFL+1))*(wnp1(:,end-1)-wn(:,end));
    % Solution
    t=t+dt;
    wnm1=wn; wn=wnp1; % Save current and previous arrays
    % Source
```

```
    wn(50,50)=10; %dt^2*20*sin(30*pi*t/20) this was the original wave source
    for i=2:nx-1, for j=2:ny-1
            wnp1(i,j) = 2*wn(i,j) - wnm1(i,j) ...
                +CFL^2*(wn(i+1,j)+wn(i,j+1)-4*wn(i,j)+wn(i-1,j)+wn(i,j-1));
        end, end
    % Check convergence
    % Visualize at selected steps
    clf;
    subplot(2,1,1);
    imagesc(x, y, wn'); colorbar; caxis([-0.2 0.02])
    title(sprintf('t = %.2f', t));
    subplot(2,1,2);
    mesh(x, y, wn'); colorbar; caxis([-0.02 0.02])
    axis([0 Lx 0 Ly -0.05 0.05]);
    shg; pause(0.01);
end
```

In the following section various figures with choices of different parameters will be presented, all of which were ran using the code above.

CHAPTER 5

NUMERICAL RESULTS

In this section images from the code will be presented and discussed.

5.1 Images

In this section frames from the code will be presented and analyzed. All snippets of the 2-D wave equation in this section have `CFL=.5`, `c=1`, `Lx=Ly=10` and `dx=.1`. Snippets chosen will be of different times, sources and reflecting vs. absorbing boundary conditions and will be labeled accordingly. Additionally, photographs of a vibrating guitar string and water drop are added so the reader can easily see the real life applications of this code.

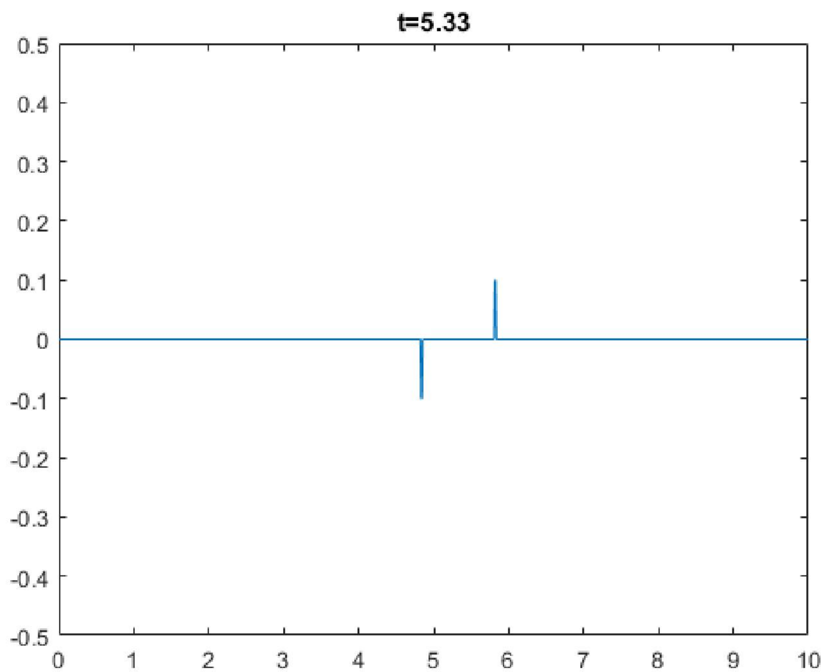

Figure 5.1: 1-D wave equation.

Figure 5.1 shows the 1-D code evaluated at `wn(49:51)=[0.1 0.2 0.1]` with `CFL=1`, `c=1` and `dx=.01`. A propagation near the center caused waves to travel in each direction.

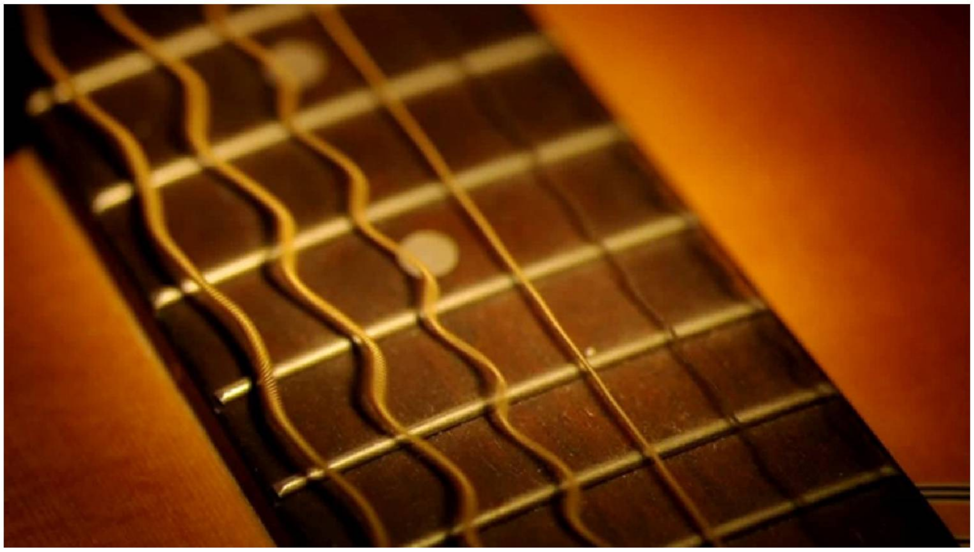

Figure 5.2: Guitar string resembling 1-D wave equation.

Figure 5.2 shows three guitar strings oscillating. Just like the 1-D snippet in figure 5.1 these waves were caused by a propagation that sent waves traveling left and right.

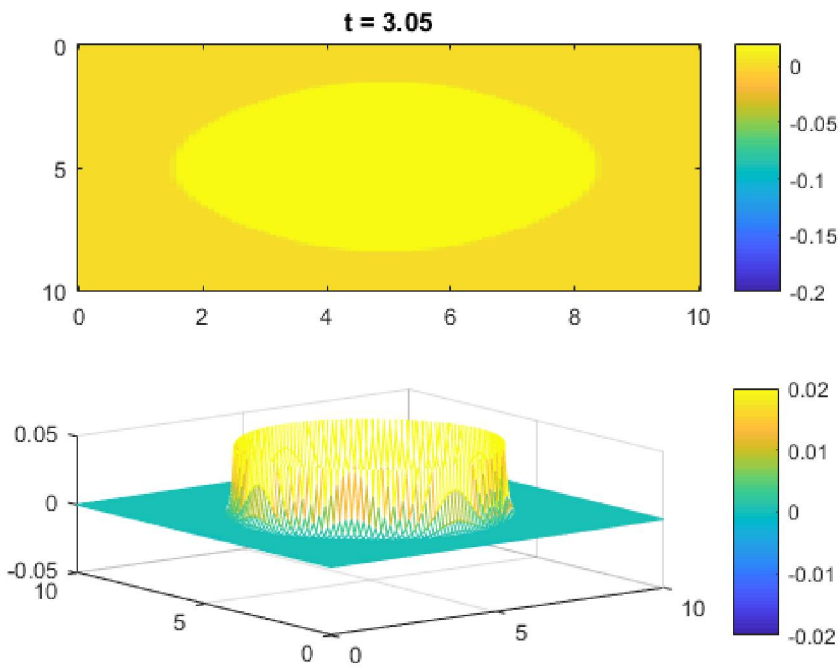

Figure 5.3: Constant propagation with absorbing boundary conditions at $t = 3.05$.

Figure 5.3 has the source defined as `wn(50,50)=10`. This snippet was chosen to show what the wave looks like after propagation and before it comes into contact with any kinds of boundaries.

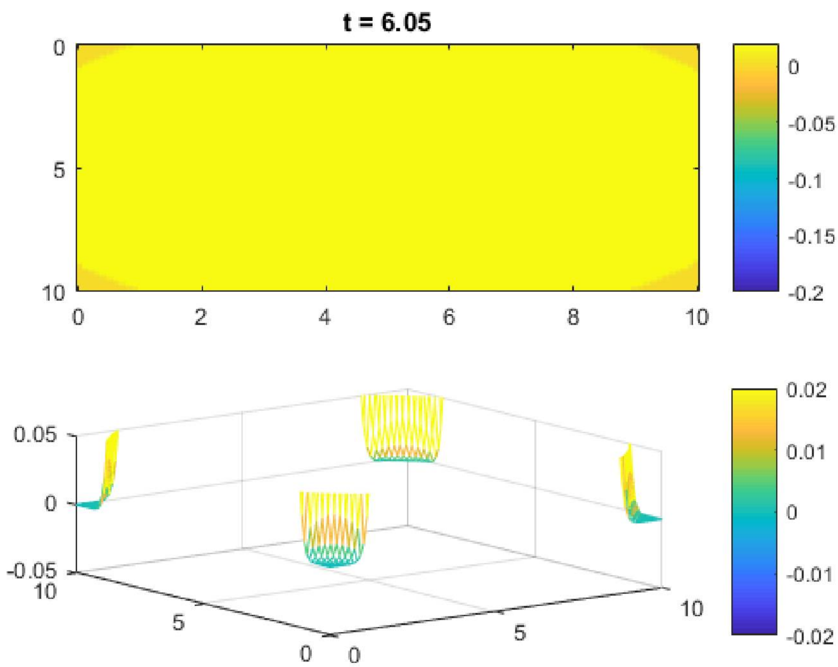

Figure 5.4: Constant propagation with absorbing boundary conditions at $t = 6.05$.

Figure 5.4 is a continuation of figure 5.3 at time 6.05. This snippet was chosen to show how absorbing boundary conditions affect the wave. We should point out that animation goes out of the plot view with respect to the z-axis. However, this does not concern us because we are most interested observing the absorbing boundary conditions.

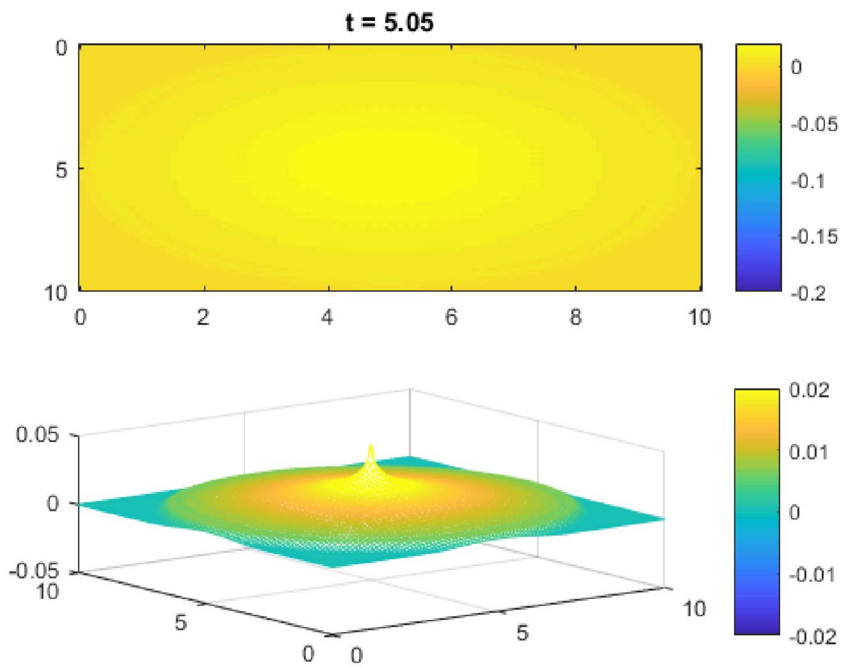

Figure 5.5: Propagation of a sin wave with absorbing boundary conditions.

Figure 5.5 is different from the previous figures because it has continual propagation of a sin wave as opposed to a single occurrence of a source. Absorbing boundary conditions are applied and this snippet was grabbed right before it comes into contact with the boundary.

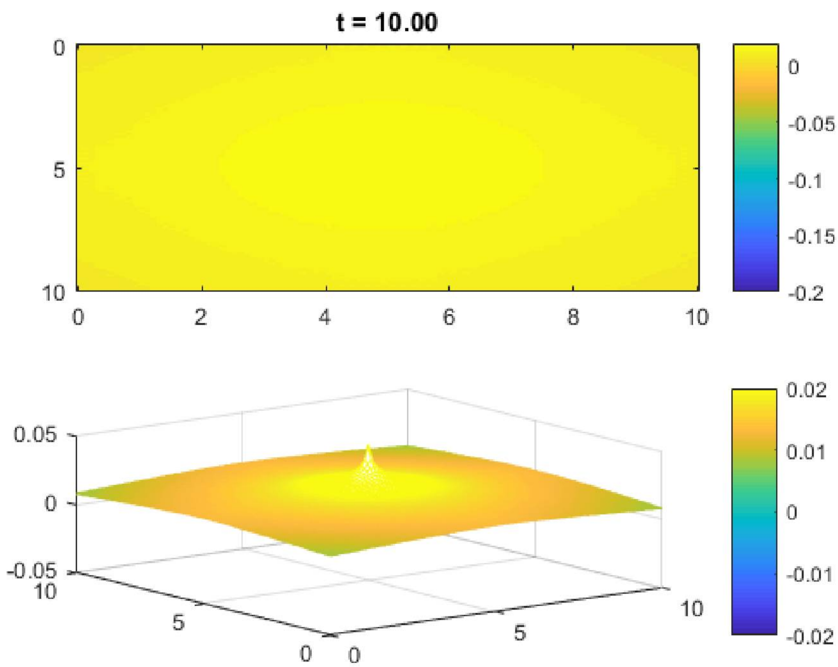

Figure 5.6: Propagation of a sin wave with absorbing boundary conditions.

Figure 5.6 is a continuation of 5.5 at time t=10. This snippet was chosen to show how sin propagation acts when it hits an absorbing boundary.

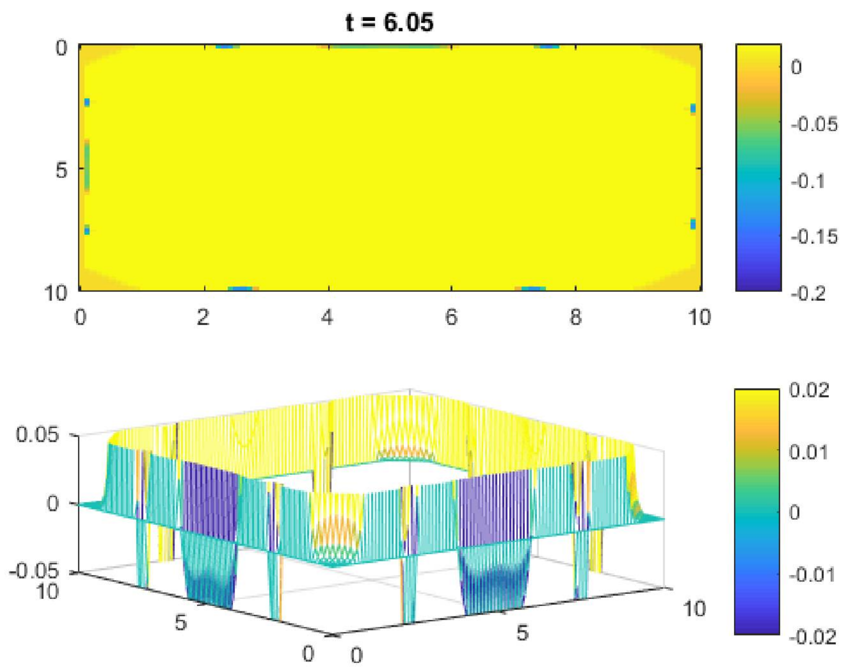

Figure 5.7: Constant propagation with Reflecting conditions t=6.05.

Figure 5.7 has the same conditions as figures 5.3 and 5.4 but this time reflecting boundary conditions were applied.

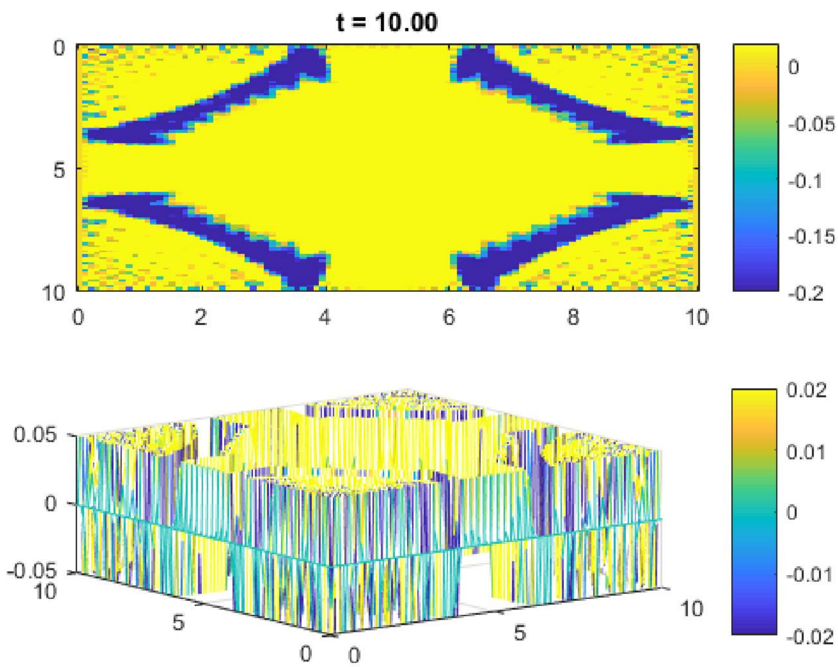

Figure 5.8: Constant propogation with reflecting conditions t=10

Figure 5.8 is a continuation of 5.7 at time t=10. This figure was chosen to show how chaotic wave reflections can become. Figures like this should make us thankful for computers that can do thousands of accurate arithmetic computations to create models we would of not been able to create 100 years ago.

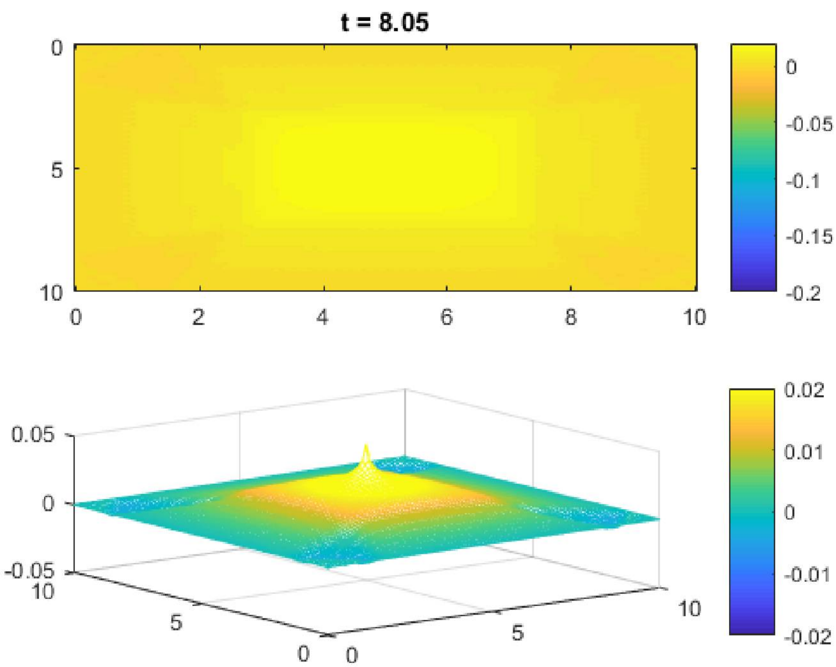

Figure 5.9: Propagation of a sin wave with reflecting boundary conditions.

Figure 5.9 has the same conditions as figures 5.5 and 5.6 but this time, reflecting boundary conditions were applied.

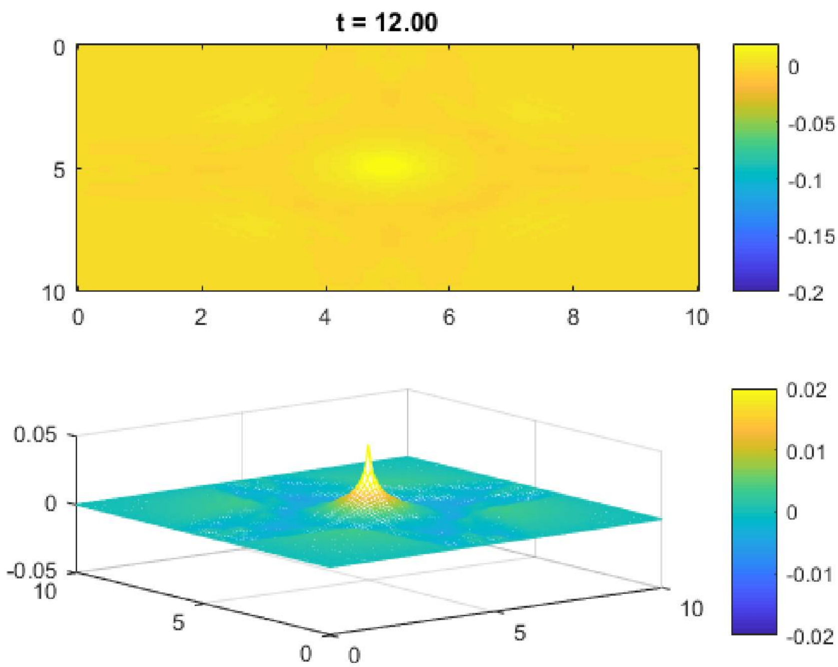

Figure 5.10: Propagation of a sin wave with reflecting boundary conditions at t=12.

Figure 5.10 is a continuation of figure 5.8 at time t=12. In this figure we can see waves reflecting off the boundary and traveling towards the source of propagation.

Figure 5.11: Water droplet resembling the 2-D wave equation

Figure 5.11 shows a water drop propagating a wave that resembles a 2-D wave equation.

Upon viewing the images from the code it is hoped that the reader has a better understanding of the goal of our numerical methods. It should now be extremely clear why it was so important to chose grid lines, establish space steps, discuss boundaries as either reflecting or absorbing, and implement a numerical scheme to represent the partial differential equation. The MATLAB figures presented were chosen to help show us the differences between choices of parameters. While the pictures of guitar string and water drop were given to compare the model to.

The ideas derived and shown in this section are desired to be a foundation for modeling 3-D ocean waves. Although ocean waves will not be modeled in this thesis they will be discussed in the following section.

CHAPTER 6

APPLICATIONS

Throughout the thesis we have stayed quite theoretical in naming $u(x, t)$. In this section of the thesis we will name a general wave function and some of its applications.

6.1 Harmonic waves

This section of the thesis will be used for becoming familiar with harmonic waves and their relationships to prepare us for a light introduction to ocean waves. The wave function we will study for this section are harmonic waves defined as

$$\eta(x, t) = A\cos\left(kx - \omega t + \phi\right). \tag{6.1.1}$$

With denotations as the following

$$\text{Amplitude}: a \quad \text{Initial pahse}: \phi \quad \text{Frequency}: f = \frac{\omega}{2\pi}$$

$$\text{Period}: T = \frac{1}{f} \quad \text{Angular frequency}: \omega \quad \text{Wave number}: k = \frac{2\pi}{\lambda}$$

$$\text{Height}: H = 2a \quad \text{Wavelength}: \lambda = \frac{2\pi}{k} \quad \text{Speed of propogation}: v$$

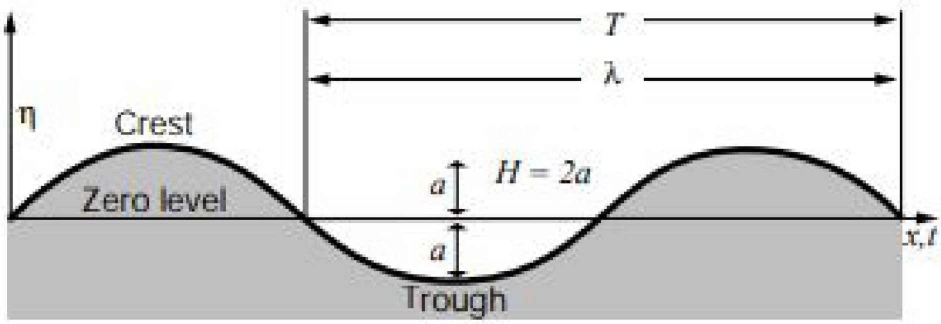

Figure 6.1: Wave profile of a harmonic wave.

The harmonic wave function represents how a 1-D wave propagates through a space and time. Although equation (6.1.1) may seem rather elementary it has many interesting features and a seemingly endless amount of applications. Before we continue much further let's recall a basic identity,

$$\text{Euler's Identity} \qquad e^{i\theta} = \cos\theta + i\sin\theta.$$

Although we are not too interested in complex waves it is important to at least mention that our harmonic wave can be written as, $\eta(x,t) = \text{RE}(Ae^{i(kx-\omega t)})$.

6.2 Ocean waves

In this section we will talk about the life cycle of an ocean wave. Particularly how wind turns into swell and how that swell reacts to bathymetry (the bottom). Although ocean waves do have three spatial variables we will consider only x and $\eta(x,t)$. We will use equation (6.1.1) to talk about waves that travel in open ocean swells. Once the wavelength becomes considerably longer than the ocean depth shallow water wave equations must be used instead.

There are plenty of sources that could cause waves in the ocean. That be it wind, Tsunami, a large ship or even an explosion. However, we will limit the discussion to wind waves. All waves of these types start from wind. The surface of the ocean can be seen as an addition of harmonic waves represented below.

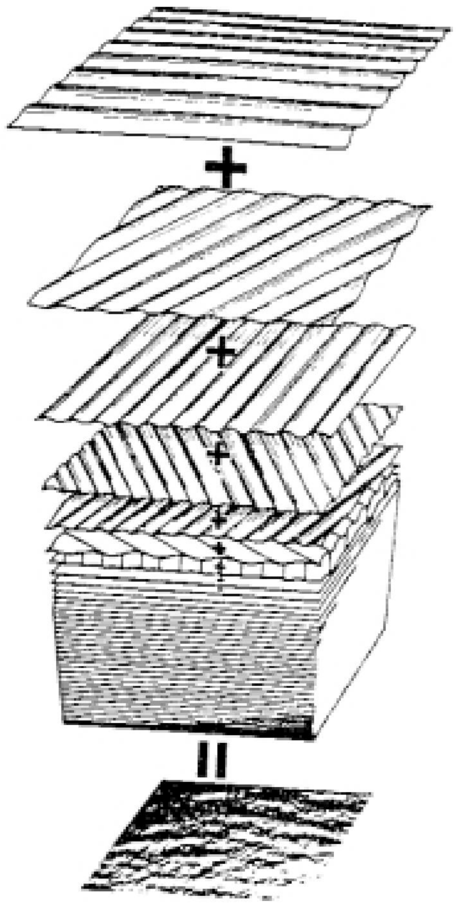

Figure 6.2: The top of the figure to the bottom shows waves added together to represent the surface condition of the ocean at the bottom.

The characteristic of the swell i.e. the wave height, frequency and direction are determined from the intensity, duration and distance of the fetch of wind. The energy from the wind is transferred into the ocean and is well preserved. These wind waves that are traveling in the same direction will now begin their journey through the ocean which we will refer to as a wave train. Waves will begin to spread out and add upon each other. This can be easily seen in the figure below.

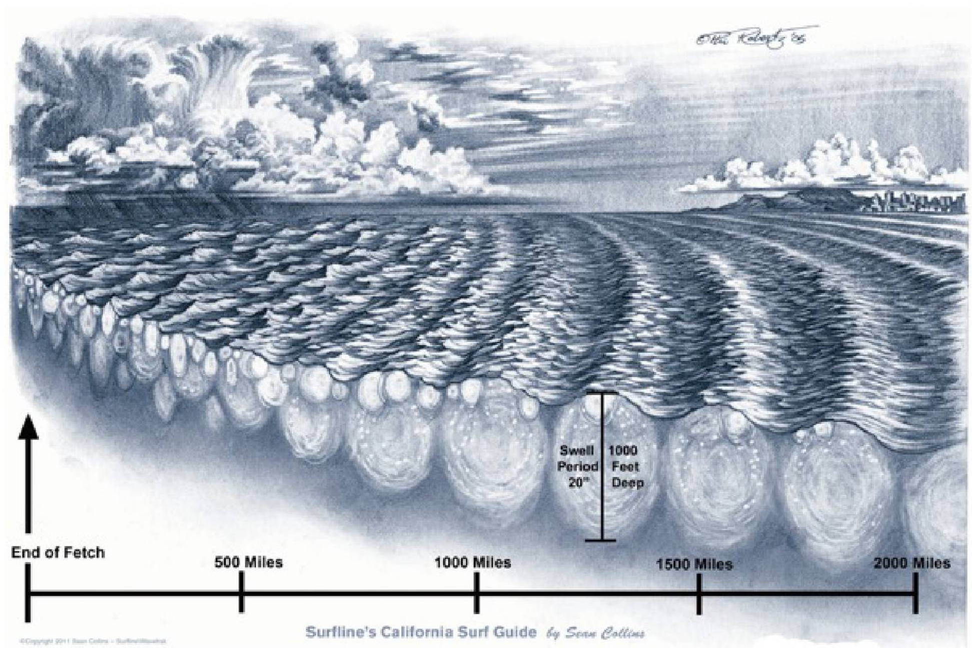

Figure 6.3: At the far left we see the storm that caused the wind, to the right we see how the waves spread out.

Now that we have a basic idea of how waves are generated let's get into the mathematics of when they approach the shoreline. Waves will travel with a velocity c and be subject to gravity $g = -9.8m/s$. Noting that $c = \omega/k$, thus we can represent lambda as

$$\lambda = cT.$$

We can use this definition of λ to solve for c as

$$c = \frac{\lambda}{T} = \frac{\omega}{k} = \sqrt{\frac{g}{k}}.$$

An extremely important idea that we have not touched upon is that at a depth approximately half the wavelength small particles can be found traveling in circles which can be seen in figure 6.4. This is an important idea and will come into effect when we talk about the depth of the ocean. We will define deep water waves as $H > \frac{\lambda}{4}$, shallow water waves as $H < \frac{\lambda}{25}$ and transitional waves as $\frac{\lambda}{4} < H < \frac{\lambda}{25}$. When waves approach the shore and come into contact with shallow water their period remains the same but their velocity decreases. Thus we will utilize the hyperbolic tangent to determine c for deep and shallow water. Where

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

From linearity theory of wave motion we will utilize

$$c^2 = \frac{g}{k} \tanh kh.$$

Notice that $\tanh kh$ in deep water will force the hyperbolic tangent close to unity and we obtain

$$c^2 = \frac{g}{k} = \frac{g\lambda}{2\pi}.$$

Similarly for shallow water we will obtain

$$c = \sqrt{gh}.$$

Another nice relation that should be mentioned is the angular frequency, which can be solved for as

$$\omega^2 = gk \tanh kh.$$

Note that this equation was not stated specifically for deep or shallow water.

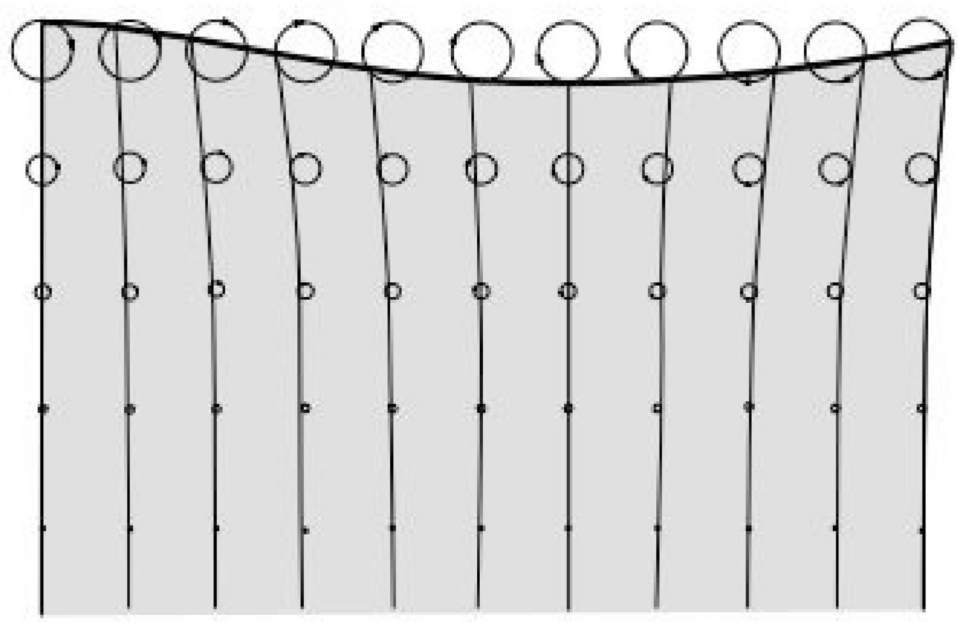

Figure 6.4: The circles represent orbital motion that particles experience at different depths.

We have now layed the ground work for the basics of how ocean swells travel. One may notice that from the above equations deeper water waves move faster. Thus, if a wave (which will be referred to as a ray) is traveling nearly parallel to a shallow depth it will begin to slow down while the portion of the wave that is still in deep water moves fast. Thus, the direction of the wave changes and refraction occurs. Refraction is displayed below in figure 6.5.

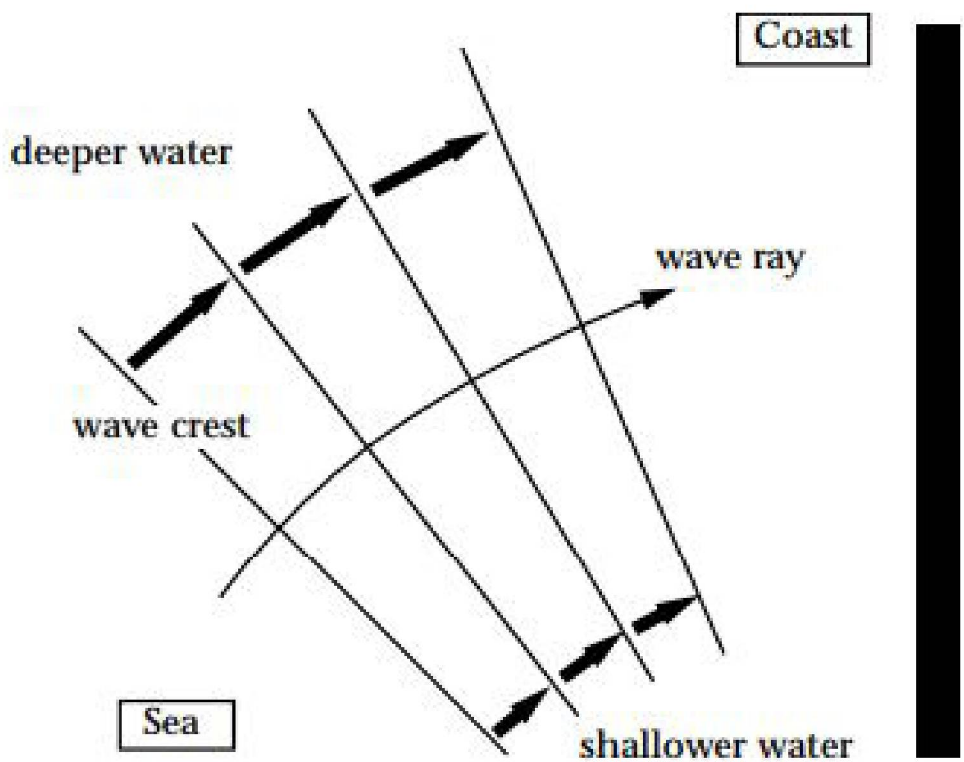

Figure 6.5: Diagram representing refractions.

In fact this is governed by Snell's law which states.

$$\frac{\sin\theta}{c_{\text{phase}}} = \text{constant}. \tag{6.2.1}$$

The angle θ is a measure between the ray and the normal to the depth contour. Another great thing to note about rays is that the energy flux between two rays is constant. Thus, if the rays refract and end up "pinching" by means of coming closer, the wave height will increase due to the energy condensed to a smaller area. Black's beach in San Diego is a great example of this. Observe the following two figures.

Figure 6.6: Refraction can be seen in the far left of the picture where the waves form a checkerboard pattern.

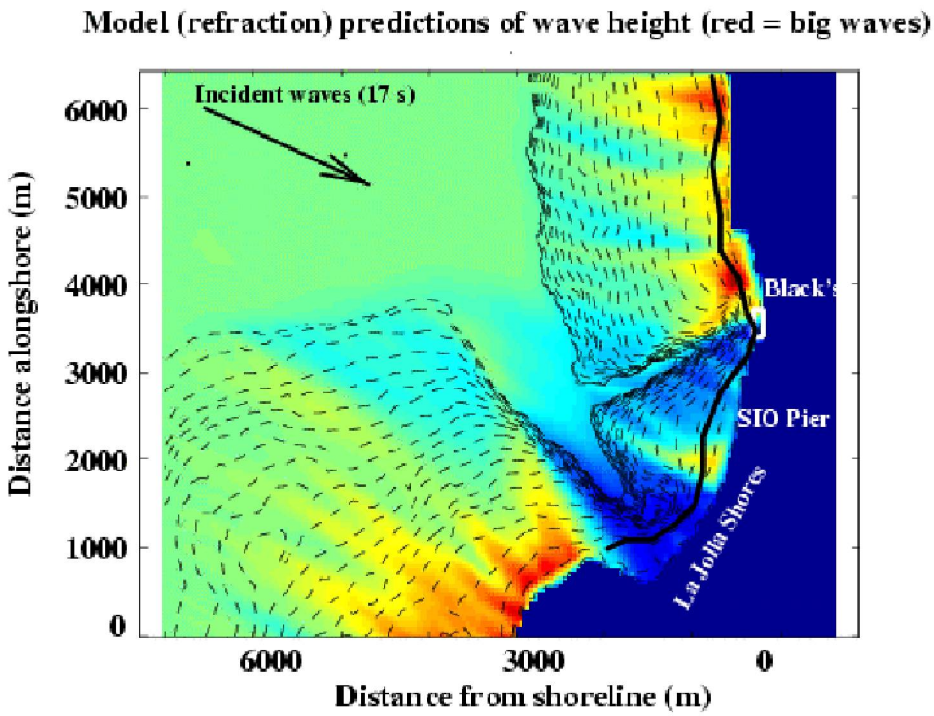

Figure 6.7: This figure shows the depths of the canyons and the forecasted wave heights. The red dot labeled blacks is the model for the figure above. This model was not ran by the author, it was obtained from [12].

Once the wave has reached shallow water and has been effected by refraction it will break when either one of three things happen;

- The crest of the wave forms an angle less than 120 degrees.
- The wave height is greater than one-seventh of the wavelength, $H > 1/7\lambda$.
- The wave height is greater than three-fourths of the water depth, $H > 3/4D$.

The height of the wave can be determined by

$$H_b = \frac{h_b}{1.28}.$$

Where h_b stands for the depth at which the wave breaks while H_B stands for the height of the breaker. To model and come up with equations once the wave breaks things get extremely complicated. In fact the Navier-Stokes existence and smoothness is a Millennium Prize Problem. Which, although would be a great thing to solve might be a tad too ambitious for a Master's thesis. Instead, in the following section we will touch upon the shallow water equations.

6.3 Shallow water equations

When representing ocean swell in deep water we found that

$$\eta(x,t) = A\cos\left(kx - \omega t + \phi\right)$$

was an appropriate model. However, once the waves arrive to shallow water we need more accurate equations that account for the bathymetry. Shallow water equations are used routinely by tsunami forecasters. Although 2-D and 3-D equations exist we will stick to the 1-D derivation for simplicity. Upon deriving the derivations we will have a better understanding of the ideas that govern fluids in shallow water. The shallow water equations in 1-D are often called the Saint-Venant equations. In this section of the thesis a derivation of the Saint-Venant equations will be presented. This derivation will utilize the physical principles of conservation of mass and conservation of linear momentum and simplifying assumptions will be stated when necessary.

For a mass balance over a control volume M the conservation of momentum states that the time rate of change of a total mass in th region of integration is equal to the net mass flux across the boundary of the region,

$$\frac{d}{dt} = \int_M \rho dV = -\int_{\partial M} (\rho v) \cdot n dA. \tag{6.3.1}$$

Where,

$$\rho = \text{fluid density}, \quad v = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \text{and} \quad n \quad \text{is the normal unit vector for} \quad \partial M.$$

By applying the divergence theorem which states

$$\int_S (\nabla \cdot F) dV = \int_{\partial V} F \cdot dA.$$

Equation (6.3.1) becomes

$$\frac{d}{dt} \int_M \rho dV = -\int_M \nabla \cdot (\rho v) dV. \tag{6.3.2}$$

We will make a simplifying assuming that ρ is smooth so we can apply the Leibniz integral rule which states

$$\frac{d}{dx}\left(\int_a^b f(x,t)dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)dt.$$

We then obtain

$$\int_M \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v)\right] dV = 0. \tag{6.3.3}$$

Thus we arrive at

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{6.3.4}$$

because M is arbitrary. Thus equation (6.3.4) is the first of the two equations in the Saint Venant Equations.

For the next equation we will utilize the conservation of linear momentum which states that the time rate of change of total momentum in a region is equal to the sum of the body and external forces on the region minus the net momentum flux across the boundary of the region.

$$\frac{d}{dt}\int_M \rho v dV = -\int_{\partial M} (\rho v) v \cdot n dA + \int_M \rho b dV + \int_{\partial M} T n dA. \tag{6.3.5}$$

Where, b: body force density per unit mass and T: Cauchy stress sensor. As we did in the derivation of our first equation let's apply the divergence theorem to equation (6.3.5) to obtain

$$\frac{d}{dt}\int_M \rho v dV = -\int_M \nabla \cdot (\rho v v) dV + \int_M \rho b dV + \int_M \nabla \cdot T dV.$$

We can subtract the terms on the right hand side and again apply Leibniz integral rule to obtain

$$\int_M \left[\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v v) - \rho b - \nabla \cdot T \right] dV = 0.$$

Again, since our region is arbitrary we can simplify it to obtain our second shallow water equation. That is,

$$\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v v) - \rho b - \nabla \cdot T = 0. \tag{6.3.6}$$

Thus written together the Saint-Venant equations are

$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0}$ for continuity and

$\boxed{\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho v v) - \rho b - \nabla \cdot T = 0.}$ for momentum.

Although we will skip the derivation, since we have come all this way we might as well state the Navier-Stokes Equations which work in 3-D. As stated many times throughout this thesis, the goal of the thesis is to dip our toes in as many areas of waves to further prepare the author in research which will certainly include the Navier-Stokes Equations.

The Navier-Stokes Equations are the following

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} + \frac{\partial(\rho u w)}{\partial z} = \frac{\partial(\tau_{xx} - \rho)}{\partial x} + \frac{\partial(\tau_{xy})}{\partial y} + \frac{\partial(\tau_{xz})}{\partial z}$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho u v)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho v w)}{\partial z}=\frac{\partial(\tau_x y)}{\partial x}+\frac{\partial(\tau_{yy}-p)}{\partial y}+\frac{\partial(\tau_{yz})}{\partial z}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho u w)}{\partial x}+\frac{\partial(\rho v w)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}=-pg+\frac{\partial(\tau_{xz})}{\partial x}+\frac{\partial(\tau_{yz})}{\partial y}+\frac{\partial(\tau_{zz}-p)}{\partial z}.$$

CHAPTER 7

CONCLUSIONS AND FUTURE WORK

This thesis concludes with an overview of what has been accomplished and the future goals that have not yet been pursued. In the conclusions section we will highlight key ideas that will hopefully serve the reader and author as fundamental ideas in more challenging problems during research. In the future work section we will talk about ideas and projects that were not attempted in this thesis.

7.1 Conclusions

Due to how many areas of science are involved in the subject matter, this thesis could have gone on seemingly forever. However, we drew a line once we arrived to the shallow water equations. We stopped here because once waves reach shallow water they will eventually end up plunging. Plunging waves are on a whole other level of research and will be studied for Phd work.

The ideas that will be taken from this thesis and used in research will be standard differential equation transformations, methods of approximation, methods of discretization, code implementations, and how to connect all of these ideas to a moving body of water. We learned that there is not just one version of the wave equation. At first, we derived an equation that was based off of time and spatial variables in the x and y plane. We then found out in section 3.5.7 that spatial variables are not just limited to x, y and z. We learned that for analytical solutions we can utilize sin and cos functions, or in other words Fourier series to model our equation. For methods of discretization we learned that approximating with polynomials, or in other words Taylor's theorem is a powerful and common approach. We will take away the ideas of finite difference and Matlab practice to help us translate equations and implement them into the computers. Finally, chapter 6 served as a great introduction to elementary ocean waves.

Aside from perhaps selfish reasons of riding waves, knowledge acquired in this thesis could be very useful to any range of professionals. Perhaps a climate scientist could use these ideas to predict the effects of glacier calving. Glacier calving is when large pieces break off from ice bergs that create displacement which could cause large waves. In fact, the believed to be largest wave ever was caused by a similar effect when an earthquake caused a landslide in Lituya Bay Alaska. The displacement was believed to have created a wave that was approximately 1700 feet high! Aside from environmental scientists knowledge of wave equations could even be used in the medical field. Ultrasound and x-rays could be improved with a better understanding of wave equations. Or maybe even wave equations could be utilized to help create artificial hearing for the deaf.

7.2 Future work

This is an exciting field to study due to how little has been accomplished relative to how much is left to discover. Like all areas of Mathematics, there is still an endless amount of questions still unanswered in wave modeling. One very important area of this thesis that was left almost completely unaddressed was the big O, the remainder of Taylor's theorem. The remainder was completely left out of the Matlab code, perhaps continuing work of this thesis would include an evaluation of error analysis.

Surfing is an emerging sport, especially in recent years we have seen an abundance of surf contests making their way into big sports networks like ESPN. Surfing will even be in the 2020 summer Olympics for the first time ever. Due to the raw elements nature of surfing, it makes it very tough to hold, film and document contests. Which is why we have seen an abundance of wave pools in recent years. Wave pools are able to produce controlled waves, unlike the fickle ocean. As Mathematicians we should at the least be interested in the Mathematics behind these very complex wave pools,wave forecasting and related control problems. Below is an image of a wave pool in Australia.

Figure 7.1: This is a wave pool in Australia that uses the "plunger" in the middle and compressed air to generate waves.

Something that is extremely exciting, unlike ocean waves which are generated by wind, wave pools are not limited to any method of wave generation. As seen in figure 7.1 one method of approach could be having an object in the center of a circular pool that moves up and down to displace water. We hope that the polar coordinates of the wave equation come to mind upon seeing this picture. Other methods that have already been implemented include sucking water into a chamber then releasing it. Also, dragging an item through the pool. All it takes to create a wave is to displace water. Who knows, maybe in the future wave pools will generate waves from underwater explosions!

Ocean waves are dramatically more challenging than 2-D wave equations. Ocean waves have a seemingly infinite amount of parameters that make them act the way they do. Wind blows over a region of water that creates little ripples that form and group together to make larger waves that travel for thousands of miles, and during their trip they are affecting by storms, reefs, shadowing etc. Although the jump from 2-D wave modeling to ocean waves is a large one, our understanding of the wave equation,harmonic motion and the shallow water equations serve as great models. A very rigorous understanding of motion in the ocean would require a study of the Navier-Stokes equation which were presented in section 6.3. The Navier-Stokes equation takes into account variables such as vorticity which were not covered in this thesis. Figure 7.2 below is presented to give the reader of what the ideal ocean wave looks like and why it would be so hard to model it.

Figure 7.2: The "Perfect" Wave. Photo taken by Rany Phenning.

Figure 7.2 shows a beautiful ocean wave, the type that engineers are trying to model in wave pools. Recall that this wave was created from wind thousands of miles away then refracted off of an underwater canyon to arrive at it's beautiful form it takes in the picture as the perfect wave. Differences between ocean waves and 2-D waves can be seen in figure 7.2. Notice that ocean waves plunge. That is, the base of the wave slows down as the crest forms upward and continues to spin. We hope that any Algebra student can recognize that this wave will not pass the vertical line test which makes it extremely difficult to write a function that could resemble it. In fact, many models for plunging waves are written as piecewise functions. Additionally, hen the crest of the wave crashes back down into the water the "explosion" causes the particle to scatter and is nearly impossible to model.

An idea that the author is interested in is a cross between a wave pool and the ocean. The idea is based off of the fact that waves can be added together to create larger waves. As we noticed with the wave pools, displacement is what it takes to create a wave. What if there was a device that could be placed in the ocean to absorb all of the small waves that are created by localized wind then "add" them together to create a larger wave. Perhaps the device could rise a weight with every wind chop then suddenly fall at the desired height to create displacement. Santa Barbara would be an area of interest for the device as it has some of the best bathymetry for surfing waves, almost always lacks large swell for surfing but constantly has wind swell which could be added together. Although the idea is very farfectched, it should not be abandoned completely. In the Gold coast of Australia, which is a premiere surf destination, $3.3 billion dollars are generated in the city annually through surf tourism alone! Surely this kind of revenue would be a great motivator for a project that combines the ideas of wave pools and ocean waves.

Continuing work of this thesis would include a rigorous transition from shallow water waves to the Navier-Stokes equation and attempts to accurately model plunging ocean waves.

List of Figures

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