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Pascal's Pyramidions and Simplexes and their Fractal Dimensions

A Thesis Presented to

The Faculty of the Mathematics Program

California State University Channel Islands

In (Partial) Fulfillment

of the Requirements for the Degree

Masters of Science

by

Kevin R. Howe

December, 2018

Signature page for the Masters in Mathematics Thesis of Kevin R. Howe

APPROVED FOR THE MATHEMATICS PROGRAM

	29 Nov. 2018
Dr. Brian Sittinger, Thesis Advisor	Date
	11/29/18
Dr. Ivona Grzegorczyk, Thesis Committee	Date
	11/29/18
Dr. Nathan Carlson, Thesis Committee	Date

APPROVED FOR THE UNIVERSITY

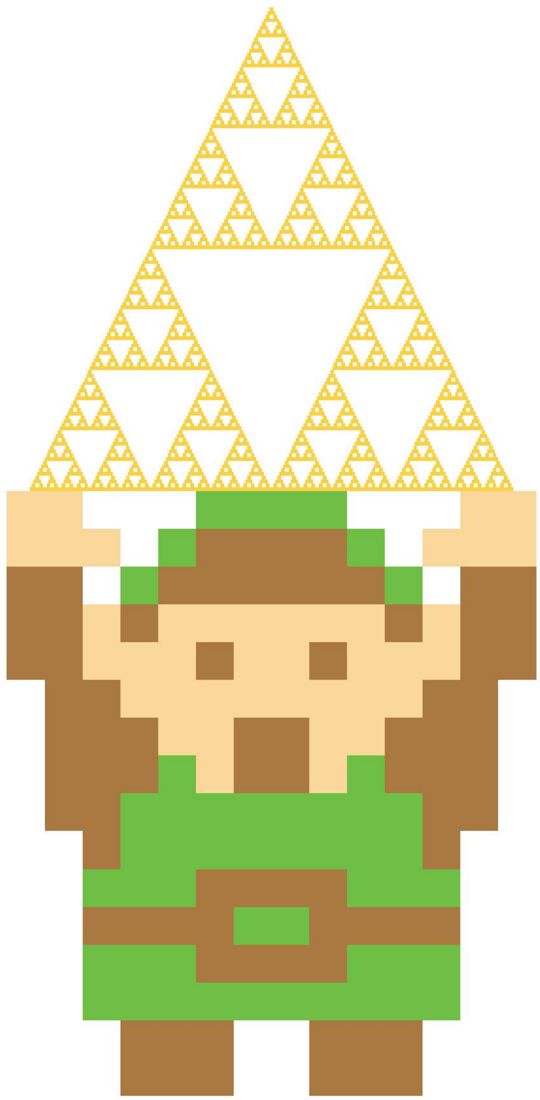
	01/28/2019
Dr. Osman Özturgut, Dean of Graduate School	Date

Acknowledgements

For all of their love and support throughout my time in graduate school, I would like to thank my family and friends. I would like to give thanks to my thesis committee members Dr. Ivona Grzegorczyk and Dr. Nathan Carlson for helping me through this process. I want to give special thanks to my thesis advisor Dr. Brian Sittinger for his help and support. I would like to give thanks to Dr. John Villalpando and Dr. Nathan Carlson for helping me research fractals as an undergraduate.

Abstract

We consider fractals generated from d-dimensional generalizations of Pascal's Triangle using modular arithmetic. We calculate their dimensions and prove formulas using Lucas' Theorem. We also study the fractal dimensions involving powers of primes by using the Box Counting Method.



*8-bit Link is the property of Nintendo, but keep it a secret to everybody.

Contents

1. Introduction

1.1. Overview.

Definition 1.1.1. *A* ***fractal*** *is a set of points in* $\mathbb{R}^d$ *with self-similarity, meaning that its parts consist of smaller copies of itself at every scale.*

Fractals are well-known for appearing in nature. Examples of natural fractals include snowflakes, coastlines, and even cauliflower. Fractals have applications in Physics to describe the patterns produced by chaos [1]. In *Applications of the Sierpinski Triangle to Musical Composition* [4], Samuel C. Dent uses fractal-generating algorithms and a configuration of note names to compose music.

Many fractals have non-integer dimensions, which distinguishes them from more familiar shapes like two-dimensional squares or three-dimensional cubes. For example, a Sierpinski Triangle embedded in $\mathbb{R}^2$ has a fractal dimension of about 1.58. Additionally, some fractals have integers as their dimensions. One example is the Sierpinski Tetrahedron in $\mathbb{R}^3$ with a fractal dimension equal to 2.

We generate fractal sets by considering the entries in Pascal's Triangle and similar subsets of Euclidean space, then reducing them with modular arithmetic. In this paper, the scaling factor p is always a prime integer. One part of this paper focuses on proving theorems about fractal dimensions by applying Lucas' Theorem. Another part considers Pascal's Triangle mod p^n, which shows self-similar patterns, but requires a more advanced method for finding fractal dimensions.

1.2. Pascal's Triangle and Generalizations. Before discussing fractals, we explain Pascal's Triangle and its generalizations. Let $d \in \mathbb{N}$ be the lowest dimension necessary to fully embed a given object in $\mathbb{R}^d$. For example, a triangle is embedded in at least two-dimensional space, and a cube is in at least three-dimensional space. In each of these Euclidean spaces, we make a generalization of Pascal's Triangle.

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \ldots$$

10 Entries in Pascal's Line

Considering a one-dimensional version of Pascal's Triangle, Pascal's Line is a sequence of numbers that starts with 1 and every entry to the right (or any arbitrary direction) is the sum of the previous 1, resulting in an infinite sequence of 1s.

$$\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \ \ldots \\
1 & 2 & 3 & 4 & 5 & \\
1 & 3 & 6 & 10 & & \\
1 & 4 & 10 & \ddots & & \\
1 & 5 & & & & \\
1 & & & & & \\
\vdots & & & & &
\end{array}$$

6 Rows of Pascal's Triangle

For $d = 2$, Pascal's Triangle is a lattice of numbers embedded in a quarter-plane that we represent as points. It starts with 1 at the top and very other entry is the sum of the two closest entries in the previous layer. In this picture, the numbers are located above and to the left. The entries can also be written as binomial coefficients (see Section 3.1). In subsequent generalizations, we omit the dots. Additionally, we do not distinguish between the lattice of numbers and the subset in $\mathbb{R}^d$ that they generate.

For $d = 3$, we create two distinct 3-dimensional versions of Pascal's Triangle.

$$\begin{array}{c} 1 \end{array} \quad
\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \quad
\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{array} \quad
\begin{array}{cccc} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{array} \quad
\begin{array}{ccccc} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{array}$$

5 Layers of Pascal's Square Pyramid

Definition 1.2.1. ***Pascal's Square Pyramid*** *is a lattice of numbers that we represent as points in a sixth of* $\mathbb{R}^3$ *with a square grid of integers at every layer. Starting with 1 at the top, every other entry is the sum of the four closest numbers from the previous layer.*

For example, $1+3+3+9$ in the fourth layer add up to the 16 in the fifth layer. The entries in the Pyramid are also products of binomial coefficients (see Section 3.2).

$$\begin{array}{ccccccccccccccccc}
1 & & 1 & 1 & & 1 & 2 & 1 & & 1 & 3 & 3 & 1 & & 1 & 4 & 6 & 4 & 1 \\
 & & 1 & & & 2 & 2 & & & 3 & 6 & 3 & & & 4 & 12 & 12 & 4 & \\
 & & & & & 1 & & & & 3 & 3 & & & & 6 & 12 & 6 & & \\
 & & & & & & & & & 1 & & & & & 4 & 4 & & & \\
 & & & & & & & & & & & & & & 1 & & & &
\end{array}$$

5 Layers of Pascal's Tetrahedron

Definition 1.2.2. ***Pascal's Tetrahedron*** *is a lattice of numbers in* $\mathbb{R}^3$ *with a triangular grid of integers at every layer. Starting with 1 at the top, every other entry is the sum of the three closest numbers in the previous layer.*

For example, $3+3+6$ in the fourth layer add up to the 12 in the fifth layer. The entries in the Tetrahedron are also the multinomial coefficients (see Section 3.3).

This paper considers two "families" of objects. Both families start with Pascal's Line and Triangle, and the families become distinct when $d \geq 3$.

Definition 1.2.3. *A* ***Pyramidion*** *is a geometric shape created by connecting a point (called the apex) to a hypercube (a generalized cube).*

This generalizes a Square Pyramid to any number of dimensions. The name "Pyramidion" comes from the capstones in Egyptian architecture [7].

Definition 1.2.4. *A* ***Simplex*** *is an d-dimensional generalization of a Tetrahedron.*

1.3. **Hausdorff Dimension.** When we reduce the entries of Pascal's Triangle with modular arithmetic, many different patterns emerge. When the modulus is a prime or a power of a prime, these patterns form self-similar structures, which generate fractals as they fill in the quarter-plane. Self-similarity means that when we zoom in or out, we see copies of the original set at different scales. We want to define what it means for these fractals to have a dimension. One common way to define the fractal dimension is through the Hausdorff dimension.

Definition 1.3.1. *The* ***Hausdorff dimension*** D *is a generalized notion of dimension which considers scaling and self-similarity. The formula* [9] *used is*

$$S^D = C \text{ or alternatively, } D = \frac{\log(C)}{\log(S)} = \log_S(C).$$

This formula only requires the scaling factor and the number of self-similar objects to find the dimension for many fractals and even familiar objects.

Example 1.3.2. *If we take a square and double the side lengths, the result is a square with 4 times the area of the original. Therefore, the Hausdorff dimension* $D = \log_2(4) = 2$*, which agrees with a square being two dimensional.*

When the formula is applied to a fractal, we only have to consider the number of self-similar shapes and the scaling factor.

1.4. Binomials and Multinomials.

Definition 1.4.1. *If we want to choose* m *objects from a set of* n *objects, the number of ways to do this is called a* ***Binomial Coefficient*** *and its value is given by*

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

Note: If $n < m$, then $\binom{n}{m}$ is equal to zero.

Definition 1.4.2. *In general, if we want to put* n *objects into* m *separate containers, then the number of ways to organize these objects is called a* ***Multinomial Coefficient*** *and its value is given by*

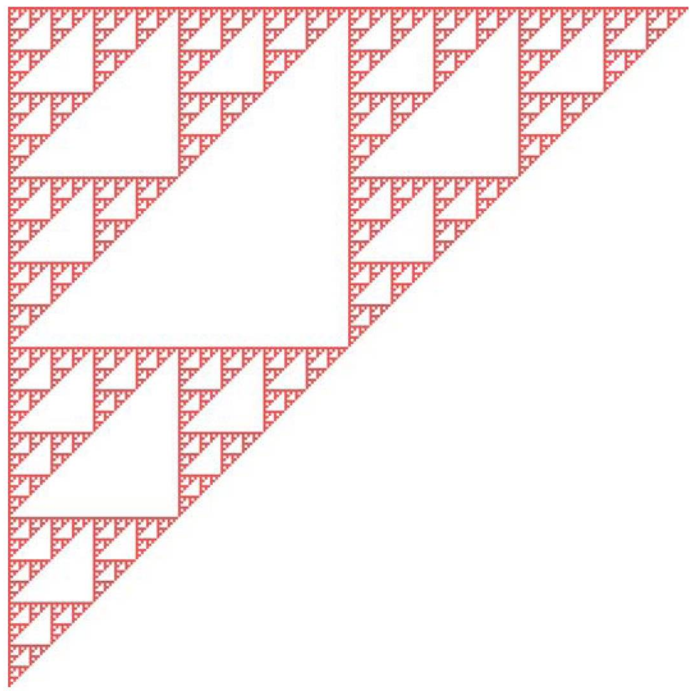
$$\binom{n}{k_1, k_2, ..., k_m} = \frac{n!}{k_1!k_2!...k_m!}$$

where $\sum_{i=1}^{m} k_i = n$.

The multinomial coefficients are more generalized than the binomial coefficients. While the binomial coefficients only have two categories, the multinomial coefficients allow m categories.

In Section 3, we prove that the entries of Pascal's Triangle and its generalizations are equal to these coefficients.

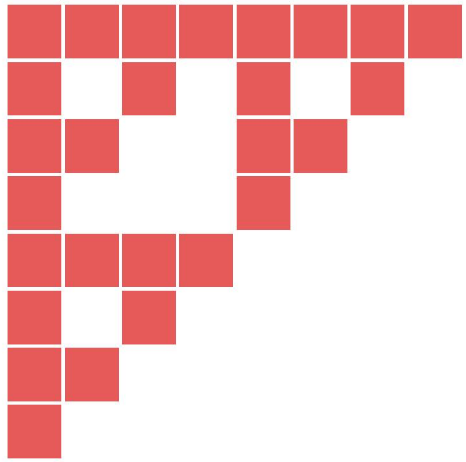
1.5. **Sierpinski Triangle.**



Sierpinski Triangle after 7 Iterations

The Sierpinski Triangle is a fractal often generated by cutting triangles out of a whole triangle and repeating this process iteratively. This fractal also resembles a pattern generated by the entries in Pascal's Triangle after reducing them mod 2. In this process, the zero entries are left blank while the remaining entries are filled in. Since Pascal's Triangle has no end point, this process constructs a Sierpinski Triangle in a quarter-plane instead of cutting pieces out of a whole triangle. In order to find the Hausdorff dimension, we can see that it has a scaling factor of 2, and the number of similar objects is 3. This means that the Hausdorff dimension is $\log_2 3 \approx 1.58$. In *Iterated Function Systems and Fractals Chaos, fractals, and noise* [5], Lasota, A. and Mackey, M. C. say that the Sierpinski Triangle can be seen by the human eye. However, in *100 years with the Sierpinski Triangle* [3], Rafael Prieto Curiel says that the Sierpinski Triangle is "invisible to the human eye, and impossible to display using the pixels of a computer screen, but certainly not beyond the scope of our imagination!" It seems that a hypothetical Sierpinski Triangle existing in the real world is not visible to the human eye. For comparison, a spider web that is very far away still has some area, but the Sierpinski Triangle has zero area.

1.6. **Box Counting Method.**



27 Boxes in 8 Rows of Pascal's Triangle mod 2

Definition 1.6.1. *The* ***Box Counting Method*** *is a technique for finding the dimension of a fractal by covering it with a grid and counting boxes that overlap with the fractal.*

For more accuracy, the boxes can be made smaller. Alternatively, the boxes can stay the same and the image of the fractal can be extended to infinity. This method is effective on fractals in Pascal's Triangle because the entries occupy discrete spaces. The Box Counting Method is also more useful than the Hausdorff dimension because it can apply to Pascal's Triangle mod p^n where p^n is a power of a prime number.

When applying this method to fractals in Pascal's Triangle, we use the formula

$$D = \frac{\log(\text{Boxes containing the fractal})}{\log(\text{Number of rows considered})}.$$

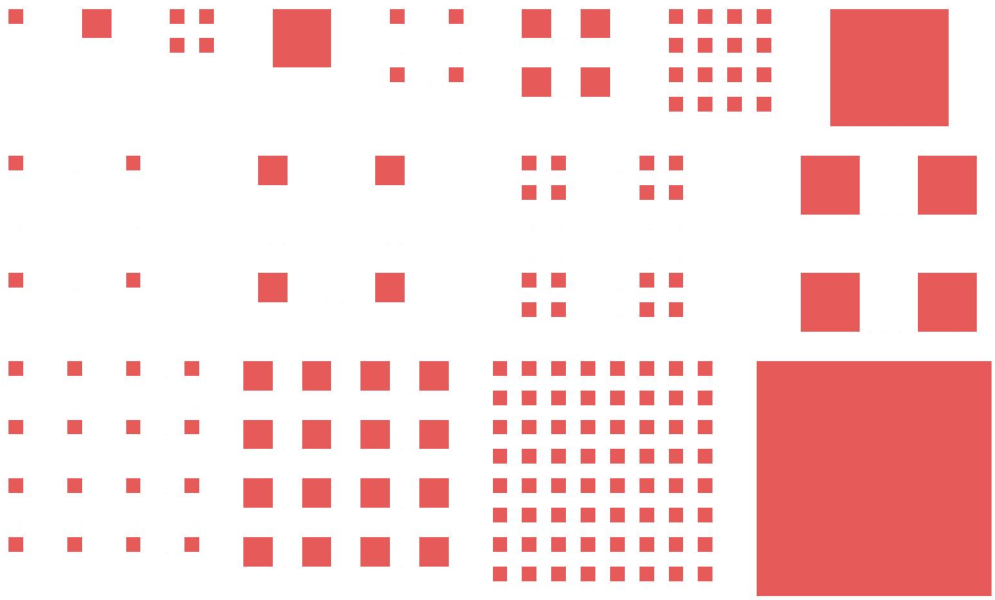
This formula also needs to end at the "bottom" of a triangle for the best results. This means that the number of boxes is equal to the number of nonzero entries, and the number of rows is always a power of p, which is also the scaling factor. For example, if we apply the Box Counting Method to 2^n rows of Pascal's Triangle mod 2, the dimensions generated are $\log_2 3$, $\log_4 9$, $\log_8 27$, ... etc. each with the same dimension.

$$\frac{\log(3^n)}{\log(2^n)} = \frac{n\log(3)}{n\log(2)} = \frac{\log(3)}{\log(2)} \approx 1.58$$

which is the same dimension as the Sierpinski Triangle.

2. Fractal Dimensions for Prime Moduli

2.1. Dimensions in Pascal's Pyramidion.



16 Layers of Pascal's Square Pyramid mod 2

Starting with Pascal's Pyramidion mod 2, we examine the dimensions of the fractals and make a general formula. In the following table, lowercase d represents the dimension of $\mathbb{R}^d$ in which the object is embedded, and capital D is the dimension of the fractal generated by the nonzero numbers after reducing them mod 2.

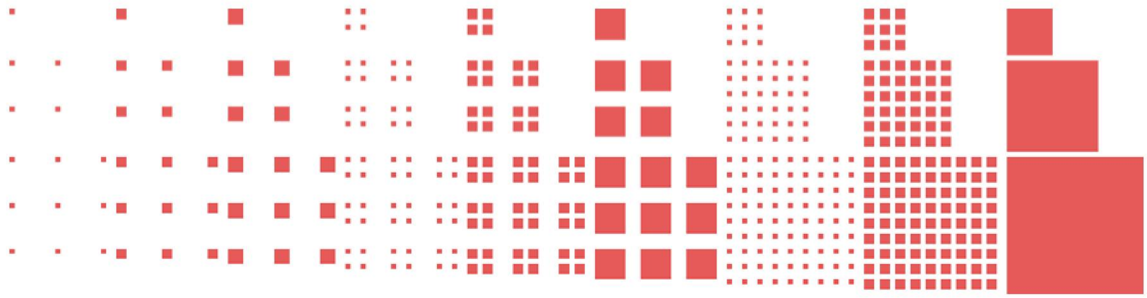
Object	d	D
Line	1	$\log_2(2) \quad = \log_2(1+1)$
Triangle	2	$\log_2(3) \quad = \log_2(1+2)$
Square Pyramid	3	$\log_2(5) \quad = \log_2(1+4)$
Cube Pyramid	4	$\log_2(9) \quad = \log_2(1+8)$

Fractal Dimensions for d-dimensional Pyramidions mod 2

Corollary 2.1.1. *The dimension of the fractal in Pascal's Pyramidion mod 2 is*

$$D = \log_2(1 + 2^{d-1}).$$

This corollary is one case of the following, more general theorem.



27 Layers of Pascal's Square Pyramid mod 3

When examining other prime moduli, we get the following tables.

Mod	Line: $d = 1$	Triangle: $d = 2$
2	$\log_2(1+1)$	$\log_2(1+2)$
3	$\log_3(1+1+1)$	$\log_3(1+2+3)$
5	$\log_5(1^0+2^0+3^0+4^0+5^0)$	$\log_5(1^1+2^1+3^1+4^1+5^1)$
Mod	Square Pyramid: $d = 3$	Cube Pyramid: $d = 4$
2	$\log_2(1+4)$	$\log_2(1+8)$
3	$\log_3(1+4+9)$	$\log_3(1+8+27)$
5	$\log_5(1^2+2^2+3^2+4^2+5^2)$	$\log_5(1^3+2^3+3^3+4^3+5^3)$

In each of these cases, the total is a sum corresponding to the modulus. For example, the table has a sum of two numbers for mod 2 and five numbers for mod 5. By rewriting them in this way, we can see that they are sums of powers of integers. This leads to a formula for the dimension of the fractal in Pascal's Pyramidion in any $\mathbb{R}^d$ and any prime modulus p.

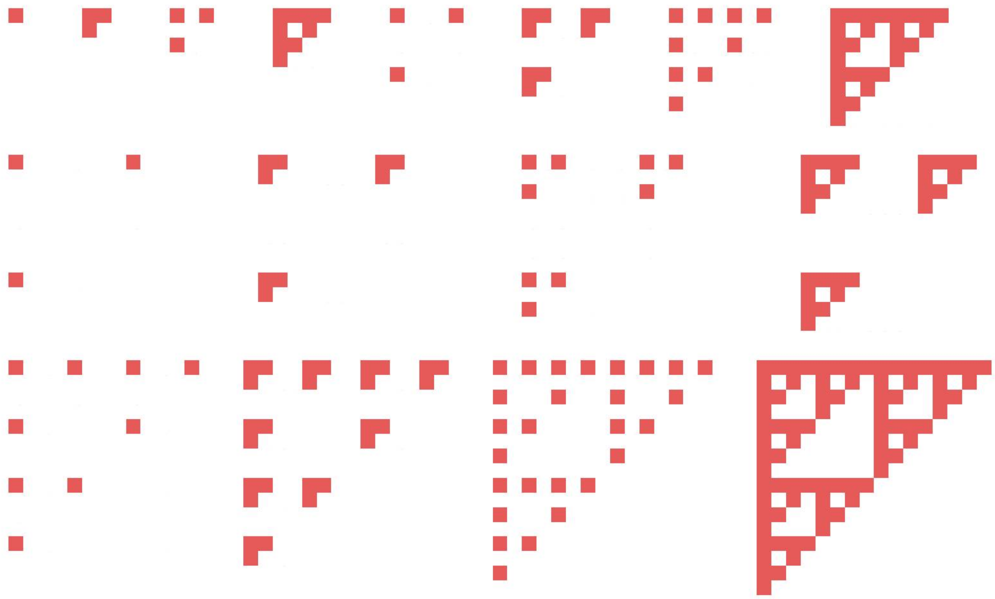
Theorem 2.1.2. *For any $d \in \mathbb{N}$ and any prime modulus p. The dimension of the fractal in Pascal's Pyramidion mod p is*

$$D = \log_p\left(\sum_{i=1}^{p} i^{d-1}\right).$$

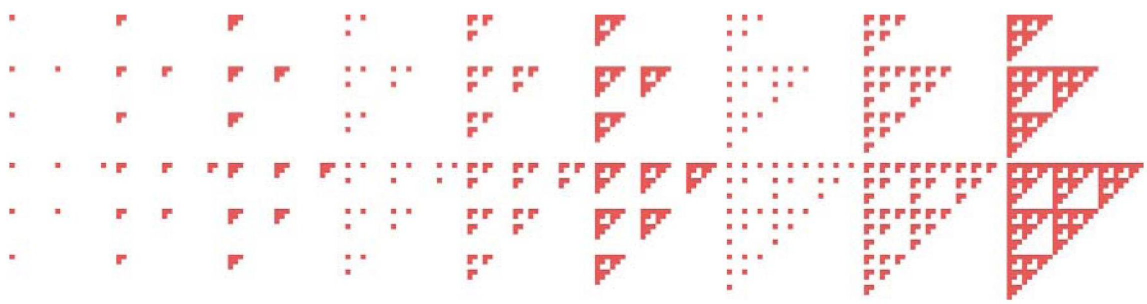
The proof of this theorem is in Section 4.3. The corollary above is a case where $p = 2$.

Next, we examine fractal dimensions in Pascal's Simplex.

2.2. **Dimensions in Pascal's Simplex.**



16 Layers of Pascal's Tetrahedron mod 2



27 Layers of Pascal's Tetrahedron mod 3

When examining prime moduli in Pascal's Simplex, we get the following tables.

Mod	Line: $d = 1$	Triangle: $d = 2$
2	$\log_2(1+1)$	$\log_2(1+2)$
3	$\log_3(1+1+1)$	$\log_3(1+2+3)$
5	$\log_5(1+1+1+1+1)$	$\log_5(1+2+3+4+5)$
Mod	Tetrahedron: $d = 3$	Hypertetrahedron: $d = 4$
2	$\log_2(1+3)$	$\log_2(1+4)$
3	$\log_3(1+3+6)$	$\log_3(1+4+10)$
5	$\log_5(1+3+6+10+15)$	$\log_5(1+4+10+20+35)$

Noticing that these numbers are sums of sums of 1s, we obtain the entries of Pascal's Triangle. By rewriting these entries as binomial coefficients, the dimension of Pascal's Simplex mod p, can be calculated from the formula in the following theorem.

Theorem 2.2.1. *For any $d \in \mathbb{N}$ and any prime modulus p. The dimension of the fractal in Pascal's Simplex mod p is*

$$D = \log_p \binom{p-1+d}{p-1} = \log_p \binom{p-1+d}{d}.$$

The proof is in Section 4.4.

Remark: In *Determining the Dimension of Fractals Generated by Pascal's Triangle* [8], Ashley Melia Reiter found and proved the exact same formula using a similar method. There is also a minor typo on the fourth page of her paper. In the proof of *Theorem 1-Multinomial Divisibility Theorem*, it says $d_i + \left(\sum_{i=1}^{k} a_2^i\right)$, but it should say d_1 instead of d_i.

2.3. **Limits of Formulas.** For our formulas for dimension, we let d be a fixed positive integer and find the limits as $p \to \infty$. Even though p needs to be a prime number, it is possible to let p go to infinity because there are infinitely many prime numbers. One general pattern is that the fractal dimension seems to be slowly approaching the dimension of the space as the modulus increases. This led to the question: if we let p go to infinity with some fixed $d \in \mathbb{N}$, do the nonzero entries "fill" the space and form a subset of $\mathbb{R}^d$ with a dimension equal to d?

Lemma 2.3.1. *Let D be the fractal dimension of the Pyramidion mod p in $\mathbb{R}^d$. Then $\lim_{p\to\infty} D = d$.*

Proof. From earlier, the formula for the Pyramidions is $D = \log_p \left(\sum_{i=1}^{p} i^{d-1}\right)$. The general formula for the sums of powers of integers, with $c \in \mathbb{N}$, is given by Faulhaber's Formula [10].

$$\sum_{k=1}^{p} k^c = \frac{1}{c+1} \sum_{m=0}^{c} \binom{c+1}{m} B_m n^{c+1-m},$$

where B_m represents the mth Bernoulli number. Putting these together, we get

$$D = \log_p\left(\frac{1}{d}\sum_{m=0}^{d-1}\binom{d}{m}B_m p^{d-m}\right) = -\log_p(d) + \log_p\left(\sum_{m=0}^{d-1}\binom{d}{m}B_m p^{d-m}\right).$$

When we let p go to infinity,

$$\lim_{p\to\infty} D = \lim_{p\to\infty}\left(-\log_p(d) + \log_p\left(\sum_{m=0}^{d-1}\binom{d}{m}B_m p^{d-m}\right)\right).$$

Since d is fixed, $\lim\limits_{p\to\infty}\log_p(d) = 0$. Factoring p^d out of the sum yields

$$\lim_{p\to\infty} D = \lim_{p\to\infty}\left(0 + \log_p\left(p^d \cdot \sum_{m=0}^{d-1}\binom{d}{m}B_m p^{-m}\right)\right).$$

Using the product rule for logs and computing the limits, we obtain

$$\lim_{p\to\infty} D = \lim_{p\to\infty}\left(d\cdot\log_p(p) + \log_p\left(\sum_{m=0}^{d-1}\binom{d}{m}B_m p^{-m}\right)\right) = d + 0 = d.$$

The last step follows from

$$\lim_{p\to\infty}\sum_{m=0}^{d-1}\binom{d}{m}B_m p^{-m} = B_0 = 1, \text{ and } \lim_{p\to\infty}\log_p(B_0) = 0.$$

□

Lemma 2.3.2. *Let D be the fractal dimension of the Simplex mod p in d-dimensional space. Then $\lim\limits_{p\to\infty} D = d$.*

Proof. Using the definition of Binomial Coefficients,

$$D = \log_p\binom{p-1+d}{p-1} = \log_p\left(\frac{(p-1+d)!}{(p-1)!d!}\right) = \log_p\left(\frac{p(p+1)(p+2)...(p-1+d)}{d!}\right)$$
$$= \log_p(p) + \log_p(p+1) + \log_p(p+2) + ... + \log_p(p-1+d) - \log_p(d!).$$

Therefore, $\lim\limits_{p\to\infty} D = 1 + 1 + 1 + ... + 1 - 0 = d.$ □

2.4. **Remark on Wolfram's paper.** In *Geometry of Binomial Coefficients* [11], Stephen Wolfram analyzes Pascal's Triangle. In the final paragraph, he talks about generalizing Pascal's Triangle to spaces with more dimensions. In the first sentence, he says "One may also consider the generalization of Pascal's triangle to a three-dimensional pyramid of trinomial coefficients." Wolfram is clearly talking about Pascal's Tetrahedron where the numbers are the trinomial coefficients. However, in the next sentence, he says "Successive rows in the triangle are generalized to planes in the pyramid, with each plane carrying a square grid of integers. The apex of the pyramid is formed from a single 1. In each successive plane, the integer at each grid point is the sum of the integers at the four neighboring grid points in the preceding plane." He goes into details about the construction of Pascal's Square Pyramid which is different from Pascal's Tetrahedron and does not contain trinomial coefficients. Later, he even states that "With $k = 2$, the fractal dimension of the pattern is $\log_2 5$." There is no doubt that he is talking about a square pyramid here because the dimension of the pattern in the square pyramid is $\log_2 5$ while the dimension of the pattern in the tetrahedron is $\log_2 4 = 2$. In the final sentence, he says "In general, the pattern obtained from the d-dimensional generalization of Pascal's triangle, reduced modulo two, has fractal dimension $\log_2(2d + 1)$." First, this does not match the previous sentence because $\log_2 5 \neq \log_2(2 \cdot 3 + 1) = \log_2 7$. Second, the actual formula is $D = \log_2(1 + 2^{d-1})$, as proven in Section 4.3.

3. Combinatorics

3.1. **Pascal's Triangle and Identity.**

$$\begin{matrix} \binom{0}{0} & \binom{1}{1} & \binom{2}{2} & \binom{3}{3} & \binom{4}{4} & \binom{5}{5} \\ \binom{1}{0} & \binom{2}{1} & \binom{3}{2} & \binom{4}{3} & \binom{5}{4} & \\ \binom{2}{0} & \binom{3}{1} & \binom{4}{2} & \binom{5}{3} & & \\ \binom{3}{0} & \binom{4}{1} & \binom{5}{2} & & & \\ \binom{4}{0} & \binom{5}{1} & & & & \\ \binom{5}{0} & & & & & \end{matrix}$$

6 Rows of Pascal's Triangle as Binomial Coefficients

We let n represent the diagonal row number and let m represent the vertical column number, with both starting at 0. When choosing m objects from n objects, the number of ways to do this is equal to the number of ways to choose $m-1$ objects from $n-1$ objects plus the number of ways to choose m objects from $n-1$ objects. The reason for this is because we can look at any object in the n objects and decide if it is chosen or not chosen. If it is chosen then there are $m-1$ objects to choose from the remaining $n-1$ objects. If it is not chosen then there are m objects to choose from $n-1$ objects. Therefore the total number of ways to choose m objects from n objects is the sum of these two numbers of ways to choose objects. This is also known as Pascal's Identity.

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}.$$

Here is another proof using the definition of binomial coefficients.

$$\begin{aligned} \binom{n-1}{m-1} + \binom{n-1}{m} &= \frac{(n-1)!}{(m-1)!(n-1-m+1)!} + \frac{(n-1)!}{m!(n-1-m)!} \\ &= (n-1)!\left[\frac{1}{(m-1)!(n-m)!} + \frac{1}{m!(n-m-1)!}\right] \\ &= (n-1)!\left[\frac{m}{m!(n-m)!} + \frac{n-m}{m!(n-m)!}\right] \\ &= \frac{n!}{m!(n-m)!} = \binom{n}{m}. \end{aligned}$$

3.2. **Binomial Coefficients in Pascal's Pyramidion.**

$$
\begin{array}{c}
1
\end{array}
\qquad
\begin{array}{cc}
1 & 1 \\
1 & 1
\end{array}
\qquad
\begin{array}{ccc}
1 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 1
\end{array}
$$

3 Layers of Pascal's Square Pyramid

$$
\begin{array}{c}
\binom{0}{0}\binom{0}{0}
\end{array}
\qquad
\begin{array}{cc}
\binom{1}{0}\binom{1}{0} & \binom{1}{0}\binom{1}{1} \\
\\
\binom{1}{1}\binom{1}{0} & \binom{1}{1}\binom{1}{1}
\end{array}
\qquad
\begin{array}{ccc}
\binom{2}{0}\binom{2}{0} & \binom{2}{0}\binom{2}{1} & \binom{2}{0}\binom{2}{2} \\
\\
\binom{2}{1}\binom{2}{0} & \binom{2}{1}\binom{2}{1} & \binom{2}{1}\binom{2}{2} \\
\\
\binom{2}{2}\binom{2}{0} & \binom{2}{2}\binom{2}{1} & \binom{2}{2}\binom{2}{2}
\end{array}
$$

3 Layers of Pascal's Square Pyramid as Binomial Coefficients

Here, we let n represent the layer in the square pyramid. In the square grid of integers in the nth layer, we let a represent the row and b represent the column, but due to symmetry, they are interchangeable. With these labels applied, any entry in the pyramid can be calculated by $\binom{n}{a}\binom{n}{b}$. To prove that these numbers are always equal to the corresponding entries in the pyramid, we need to show that these numbers share the two defining characteristics of Pascal's Square Pyramid. The first characteristic is that it starts with 1 at the top. The second is that each number is equal to the sum of the four numbers above it. These are the defining characteristics of Pascal's Square Pyramid, hence showing that they also apply to products of binomial coefficients proves that the two pyramids are equal.

Theorem 3.2.1. *In Pascal's Square Pyramid, the entry in layer n, row a, and column b is equal to*

$$\binom{n}{a}\binom{n}{b}.$$

Proof. We need to check two conditions. First, that the number at the top is 1, and second, that every other number is the sum of the four numbers above it. The first number in the pyramid is equal to $\binom{0}{0}\binom{0}{0} = 1$.

Next, any arbitrary number in layer n, row a, column b is equal to the four numbers above it. These four numbers are all in layer $n-1$ and they are located in the rows $a-1$ and a, and the columns $b-1$ and b.

The sum of the previous four entries is

$$\binom{n-1}{a-1}\binom{n-1}{b-1}+\binom{n-1}{a}\binom{n-1}{b-1}+\binom{n-1}{a-1}\binom{n-1}{b}+\binom{n-1}{a}\binom{n-1}{b}.$$

We can rewrite this as a product of sums

$$\left(\binom{n-1}{a-1}+\binom{n-1}{a}\right)\cdot\left(\binom{n-1}{b-1}+\binom{n-1}{b}\right).$$

Then, using Pascal's Identity on these two sums, we get the final result

$$\binom{n}{a}\binom{n}{b}.$$

□

Next, we examine the general cases with d dimensions.

Theorem 3.2.2. *In a d-dimensional Pyramidion, the entry in layer n with coordinates* $(a_1, a_2, ..., a_{d-1})$, *is equal to*

$$\binom{n}{a_1}\binom{n}{a_2}...\binom{n}{a_{d-1}}.$$

Proof. First, the number at the top is $\binom{0}{0}\binom{0}{0}...\binom{0}{0}=1$. Next, any arbitrary entry is the sum of the 2^{d-1} entries in the previous layer. Like before, these entries are located at a_1-1 and a_1, up to $a_{d-1}-1$ and a_{d-1}.

The sum of 2^{d-1} entries equals the product of sums

$$\prod_{i=1}^{d-1}\left[\binom{n-1}{a_i-1}+\binom{n-1}{a_i}\right].$$

Using Pascal's Identity on each of these sums, we get the product

$$\binom{n}{a_1}\binom{n}{a_2}...\binom{n}{a_{d-1}}.$$

□

Next, we examine Pascal's Simplex.

3.3. **Multinomials in Pascal's Simplex.**

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 2 & 1 & 1 & 3 & 3 & 1 \\
 & 1 & & 2 & 2 & & 3 & 6 & 3 & \\
 & & & 1 & & & 3 & 3 & & \\
 & & & & & & 1 & & &
\end{array}$$

4 Layers of Pascal's Tetrahedron

$$\begin{array}{cccccccccc}
\binom{0}{0,0,0} & \binom{1}{1,0,0} & \binom{1}{0,1,0} & \binom{2}{2,0,0} & \binom{2}{1,1,0} & \binom{2}{0,2,0} & \binom{3}{3,0,0} & \binom{3}{2,1,0} & \binom{3}{1,2,0} & \binom{3}{0,3,0} \\
 & \binom{1}{0,0,1} & & \binom{2}{1,0,1} & \binom{2}{0,1,1} & & \binom{3}{2,0,1} & \binom{3}{1,1,1} & \binom{3}{0,2,1} & \\
 & & & \binom{2}{0,0,2} & & & \binom{3}{1,0,2} & \binom{3}{0,1,2} & & \\
 & & & & & & \binom{3}{0,0,3} & & &
\end{array}$$

4 Layers of Pascal's Tetrahedron as Multinomial Coefficients

We let n represent the layer in the tetrahedron. In the triangular grid of integers in the nth layer, we no longer have rows and columns to measure locations. Instead, we let a represent the top left point, with the value of a decreasing as the distance from this point increases. We let b represent the top right point and have the value of b decrease as we move away from the top right. (This corresponds to columns, but only because of the angle of the figure above.) We let c represent the bottom left corner of the triangle and the value of c decreases as we move upward. (This corresponds to rows because of the angles.) Additionally, since $a+b+c=n$, we can use two values to find the third value. Also, due to symmetry, the three letters are interchangeable. With these labels applied, any entry in the tetrahedron can be calculated by $\binom{n}{a,b,c}$. To prove that these numbers are always equal to the corresponding entries in the tetrahedron, we need to show that these numbers share the two defining characteristics of Pascal's Tetrahedron. The first characteristic is that it starts with 1 at the top. The second is that each number is equal to the sum of the three numbers above it arranged in an inverted triangle. These are the defining characteristics of Pascal's Tetrahedron, hence proving that they also apply to multinomial coefficients shows that the two tetrahedrons are equal.

Theorem 3.3.1. *In Pascal's Tetrahedron, the entry in layer n, with triangular coordinates (a, b, c), is equal to*

$$\binom{n}{a, b, c}.$$

Proof. We need to check two conditions. First, that the number at the top is 1, and second, that every other entry is the sum of the three numbers above it. The first number in the pyramid is equal to $\binom{0}{0,0,0} = 1$.

Next, any arbitrary entry in layer n, triangular coordinates a, b, and c should be equal to the sum of the three numbers above it. These numbers are all in layer $n - 1$ and each entry has 1 subtracted from one of the coordinates.

The sum of the three previous entries is

$$\binom{n-1}{a, b, c-1} + \binom{n-1}{a, b-1, c} + \binom{n-1}{a-1, b, c}.$$

Rewriting them as fractions, we obtain

$$\frac{(n-1)!}{a!b!(c-1)!} + \frac{(n-1)!}{a!(b-1)!c!} + \frac{(n-1)!}{(a-1)!b!c!}.$$

Multiplying to get $a!b!c!$ as a common denominator, we obtain

$$\frac{c(n-1)!}{a!b!c!} + \frac{b(n-1)!}{a!b!c!} + \frac{a(n-1)!}{a!b!c!}.$$

Since $a + b + c = n$, the expression simplifies to

$$\frac{(a+b+c)(n-1)!}{a!b!c!} = \frac{n!}{a!b!c!} = \binom{n}{a, b, c}.$$

□

Theorem 3.3.2. *In Pascal's d-dimensional simplex, the entry in layer n with coordinates $(a_1, a_2, ..., a_d)$ is equal to*

$$\binom{n}{a_1, a_2, ..., a_d}.$$

Proof. First, the number at the top is $\binom{0}{0,0,...,0} = 1$. Next, any arbitrary entry should be equal to the sum of the d entries in the previous layer. These entries are located in layer $n - 1$, and each is one coordinate away from the chosen entry.

The sum of the previous entries is

$$\binom{n-1}{a_1-1, a_2, ..., a_d} + \binom{n-1}{a_1, a_2-1, ..., a_d} + ... + \binom{n-1}{a_1, a_2, ..., a_d-1}.$$

Rewriting as fractions yields

$$\frac{(n-1)!}{(a_1-1)!a_2!...a_d!} + \frac{(n-1)!}{a_1!(a_2-1)!...a_d!} + ... + \frac{(n-1)!}{a_1!a_2!...(a_d-1)!}.$$

Multiplying to create a common denominator, we obtain

$$\frac{a_1(n-1)!}{a_1!a_2!...a_d!} + \frac{a_2(n-1)!}{a_1!a_2!...a_d!} + ... + \frac{a_d(n-1)!}{a_1!a_2!...a_d!}.$$

Since $a_1 + a_2 + ... + a_d = n$, the expression simplifies to

$$\frac{(a_1 + a_2 + ... + a_d)(n-1)!}{a_1!a_2!...a_d!} = \frac{n!}{a_1!a_2!...a_d!} = \binom{n}{a_1, a_2, ..., a_d}.$$

□

This shows that the generalizations of Pascal's triangle have binomials and multinomials as their entries. In the next section, we apply Lucas' Theorem to prove several results.

4. Lucas' Theorem and Resulting Proofs

4.1. **Lucas' Theorem and Corollaries.** Lucas' Theorem is a fact about binomial coefficients that involves the base p expansion of n and m. It shows that the binomial coefficient of n and m is congruent to the product of binomial coefficients n_i and m_i mod p.

Theorem 4.1.1 (Lucas' Theorem [6])**.** *Let p be a prime, and let*

$$n = n_0 + n_1p + n_2p^2 + \cdots + n_kp^k,$$

$$m = m_0 + m_1p + m_2p^2 + \cdots + m_kp^k,$$

where $0 \leq n_i < p$ and $0 \leq m_i < p$. Then

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\binom{n_2}{m_2}\cdots\binom{n_k}{m_k} \quad \text{mod } p.$$

Corollary 4.1.2. *Using the notation in Lucas' Theorem,*

$$\binom{n}{m} \equiv 0 \text{ mod } p \text{ if and only if } m_i > n_i \text{ for some } i \in \{0, 1, ..., k\}.$$

Proof. ($\Leftarrow$) If there exists an $i \in \{0, 1, ..., k\}$ where $m_i > n_i$, then $\binom{n_i}{m_i} = 0$. When this is multiplied by the other binomial coefficients in Lucas' Theorem, the product becomes zero and $\binom{n}{m} \equiv 0 \mod p$.

($\Rightarrow$) If we suppose that for every i, $m_i \leq n_i$, then

$$\binom{n_i}{m_i} = \frac{n_i!}{m_i!(n_i - m_i)!}.$$

Since every $n_i < p$ and p is prime, $p \nmid n_i!$ for any n_i. Therefore p does not divide their product and $p \nmid \binom{n}{m}$. Equivalently, $\binom{n}{m} \not\equiv 0 \mod p$. □

Corollary 4.1.3 (Anton's Lemma [2])**.** *If $n, m < p^k$, then for all $x, y \geq 0$,*

$$\binom{n + x \cdot p^k}{m + y \cdot p^k} \equiv \binom{n}{m}\binom{x}{y} \quad \text{mod } p.$$

Proof. We suppose that $n, m < p^k$ and $x, y \geq 0$. Since n and m are both less than p^k, we know that n and m have the base p representations $n = n_0 + n_1p + n_2p^2 + ... + n_{k-1}p^{k-1}$ and

$m = m_0 + m_1 p + m_2 p^2 + ... + m_{k-1} p^{k-1}$. Therefore we can add $x \cdot p^k$ to n and $y \cdot p^k$ to m and use Lucas' Theorem to find that

$$\binom{n + x \cdot p^k}{m + y \cdot p^k} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\binom{n_2}{m_2} \cdots \binom{n_{k-1}}{m_{k-1}}\binom{x}{y} \equiv \binom{n}{m}\binom{x}{y} \mod p.$$

□

Next, we use these corollaries to prove many facts about Pascal's Triangle and its generalizations. The first corollary can be used to prove the existence of zeros in these objects and Anton's Lemma can be used to prove that entries are nonvanishing when reduced modulo p.

4.2. **Fractals in Pascal's Triangle.**

Theorem 4.2.1. *For any prime modulus p, the dimension of the fractal in Pascal's Triangle mod p is*

$$D = \log_p \left(\frac{p(p+1)}{2} \right).$$

Proof. We want to show that the fractal given in p^{n+1} rows of Pascal's Triangle mod p contains exactly $\frac{p(p+1)}{2}$ copies of the fractal given in p^n rows, with inverted triangles of zeros between them. We begin by indexing Pascal's triangle with rows and columns using r for the row number and c for the column number such that any element in Pascal's Triangle is given by $\binom{r}{c}$. If we consider a triangle existing in p^n rows (going from 0 to $p^n - 1$) with the restrictions that $0 \leq r < p^n$ and $0 \leq c \leq r$, then Anton's Lemma implies

$$\binom{r}{c} \equiv \binom{r + x \cdot p^n}{c + y \cdot p^n} \mod p$$

for all $0 \leq x < p$ and $0 \leq y \leq x$. This means that the top triangle with rows r and columns c is equivalent to the triangles located in rows $r + x \cdot p^n$ and columns $c + y \cdot p^n$. Each pair of x and y correspond to one triangle in p^{n+1} rows, hence the total number of triangles is equal to the number of pairs of x and y. When $x = 0$, y is restricted by x, implying that $y = 0$ which represents the top triangle. When $x = 1$, then y can be 0 or 1, resulting in two triangles after the first. As x increases, the total becomes a sum of increasing integers, also known as the triangular numbers. With x starting at 0 and ending at $p - 1$, we get a sum of integers from 1 to p. The result is exactly $\frac{p(p+1)}{2}$ copies of the top triangle given in p^n rows.

Next, we need to show that the entries in between the triangles are made up entirely of zeros. The inverted triangles of zeros correspond to the entries located at $\binom{r+x\cdot p^n}{c+y\cdot p^n}$ where $r < c < p^n$. When applying Anton's Lemma, we get

$$\binom{r+x\cdot p^n}{c+y\cdot p^n} \equiv \binom{r}{c}\binom{x}{y} \equiv 0 \cdot \binom{x}{y} \equiv 0 \mod p$$

because $c > r$ in these entries.

With the number of copies and the scaling factor known, we can apply the formula for the Hausdorff dimension to find that $D = \log_p\left(\frac{p(p+1)}{2}\right)$. □

4.3. **Fractals in Pascal's Pyramidion.**

Theorem 4.3.1. *For any $d \in \mathbb{N}$ and any prime modulus p, the dimension of the fractal in Pascal's Pyramidion mod p is*

$$D = \log_p\left(\sum_{i=1}^{p} i^{d-1}\right).$$

Proof. From 3.2, the entry in layer n with coordinates a_1, a_2, ..., a_{d-1}, is equal to

$$\binom{n}{a_1}\binom{n}{a_2}\cdots\binom{n}{a_{d-1}}.$$

Each entry is nonzero if and only if each coefficient is nonzero mod p.

We need to show that the shape in p^{k+1} layers contains exactly $1 + 2^{d-1} + 3^{d-1} + ... + p^{d-1}$ copies of the shape in p^k layers. Here, the word "copy" does not mean that the entries are exactly the same, only that they form the same shape of nonzero entries. One copy exists because p^{k+1} layers contain the first p^k layers. Since the first p^n layers are numbered from 0 to $p^k - 1$, we know that the layer number n is restricted to $0 \le n < p^k$. Each of the other indexing numbers are limited by n, hence $0 \le a_i \le n$ with i ranging from 0 to $d-1$. Next, Anton's Lemma implies that for all $0 \le x_i < p$ and $0 \le y_i \le x_i$,

$$\binom{n + x_i\cdot p^k}{a_i + y_i\cdot p^k} \equiv \binom{n}{a_i}\binom{x_i}{y_i} \mod p.$$

This means that each pair of x and y correspond to one copy of the shape in p^k layers. When x and y are both equal to zero, we get the first shape. When x is equal to 1, then y can be either 0 or 1, giving us 2 options for each i. Since $1 \le i \le d-1$, the number of copies between p^k rows and $2 \cdot p^k$ rows is exactly 2^{d-1}. Next, when x is equal to 2, the number of copies corresponds to 3^{d-1} since there are three options for y. Similarly, for every $x < p$, there is a corresponding number of options for y in each coordinate, resulting in another copy. The sum of these nonzero copies is $\sum_{i=1}^{p} i^{d-1}$.

We also know that the spaces in between these copies contain zeros. Since the outermost surface of the Pyramidion is Pascal's Triangle, the zero entries from Pascal's Triangle make the entire product of binomials equal to zero. The zeros in Pascal's Triangle mod p poke colloquial holes that go all the way through the fractal.

With the number of copies and the scaling factor, we apply the formula to find the Hausdorff dimension $D = \log_p \left(\sum_{i=1}^{p} i^{d-1} \right)$. □

The following corollary relates to the remarks about Wolfram's paper in Section 2.4.

Corollary 4.3.2. *The formula for the dimension of d-dimensional Pyramidions mod 2 is* $D = \log_2(1 + 2^{d-1})$.

4.4. **Fractals in Pascal's Simplex.**

Theorem 4.4.1. *For any $d \in \mathbb{N}$ and any prime modulus p, the dimension of the fractal in Pascal's Simpex is*

$$D = \log_p \binom{p+d-1}{p-1} = \log_p \binom{p-1+d}{d}.$$

Proof. From 3.3, the entry in layer n with coordinates a_1, a_2, ..., a_d is equal to

$$\binom{n}{a_1, a_2, ..., a_d}.$$

We need to show that the shape given in p^{k+1} layers contains exactly $\binom{p+d-1}{p-1}$ copies of the shape in p^k layers. Here, "copy" means that the nonzero entries form the same shape, not that the entries are equivalent mod p. The first p^k layers are numbered from 0 to $p^k - 1$, with $0 \leq n < p^k$. All of the other coordinates are limited by n, with $a_1 + a_2 + ... + a_d = n$ and $0 \leq a_i$. This restriction implies that, when considering the next set of layers, we must add $x \cdot p^k$ to both sides of the equation. Therefore $a_1 + a_2 + ... + a_d + x \cdot p^k = n + x \cdot p^k$ with $0 \leq x < p$. Each x corresponds to a set of layers containing some number of copies.

We begin by examining values for x, which leads to an induction. When $x = 0$, we get the original copy given in p^k layers. There is only one choice here, equal to $\binom{d}{0}$. When $x = 1$, the layer numbers go up from n to $n + p^k$. Since the layer number is also equal to the sum of the a_i, we need to add p^k to one of the a_i, and there are $\binom{d}{1}$ options. Adding the $x = 0$ case with the $x = 1$ case, we get $\binom{d}{0} + \binom{d}{1} = \binom{d+1}{1}$ When $x = 2$, we have the next set of layers with $2p^k$ added to the sum of the a_i. When distributing this, we have the option to add the $2p^k$ to each a_i, giving us d choices. We also have the option to give only one of the two p^k to one a_i and adding the other p^k to a different a_j. This gives us $\binom{d}{2}$ options. Putting these together gives us $\binom{d}{1} + \binom{d}{2} = \binom{d+1}{2}$ options when $x = 2$. Adding the $x = 2$ case to the previous cases gives us $\binom{d+1}{1} + \binom{d+1}{2} = \binom{d+2}{2}$. When $x = 3$, we have three p^k to distribute across the a_i. The number of ways to do this is the sum of the number of ways to add all three to one, adding one to one and two to another, and all three separate. With the first case, we have $\binom{d}{1}$ options. In the second, we have $\binom{d}{2}$ options, multiplied by 2, since we can give either the p^k or the $2p^k$ to each of the ones we pick. In the third, we have $\binom{d}{3}$ options. The total for these is $\binom{d}{1} + \binom{d}{2} + \binom{d}{2} + \binom{d}{3}$. This adds up to $\binom{d+1}{2} + \binom{d+1}{3} = \binom{d+2}{3}$. Adding the total from here to the cumulative total gives us $\binom{d+2}{2} + \binom{d+2}{3} = \binom{d+3}{3}$.

Next, we use induction on x to find the cumulative total at the end. We suppose that the cumulative total is currently $\binom{d+x-1}{x-1}$. For any value of x, the number of p^ks we need to distribute is equal to x, and we have d places to put them. This is a case of distributing x identical objects among d groups. This has a total of $\binom{d+x-1}{d-1} = \binom{d+x-1}{x}$. If we add this number to the cumulative total, we get $\binom{d+x-1}{x-1} + \binom{d+x-1}{x} = \binom{d+x}{x}$.

In general, the total sum is equal to $\binom{d+x}{x}$ with x increasing with each set of layers. Since x is limited by $x < p$, the total for all of these copies is $\binom{d+p-1}{p-1}$.

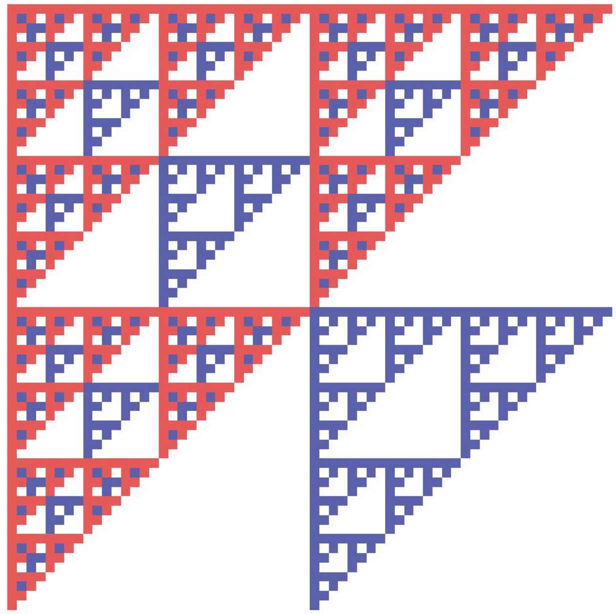
In the gaps between these copies, the p^ks are broken into smaller pieces distributed among the a_i. The resulting entries are then reduced to zero mod p.

Using the number of copies and the scaling factor, we compute the dimension of the fractal.

$$D = \log_p \binom{p+d-1}{p-1} = \log_p \binom{p-1+d}{d}.$$

$\square$

5. Powers of Primes



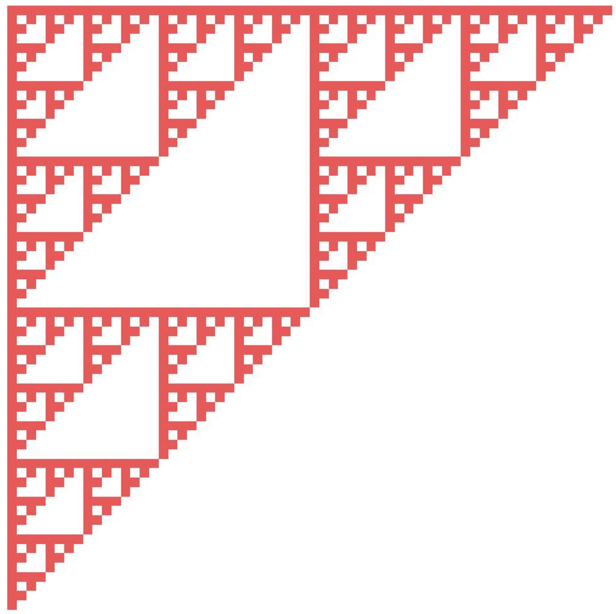
Pascal's Triangle mod 4

Pascal's Triangle mod 4 creates a pattern which resembles the Sierpinski Triangle, but with another Sierpinski Triangle inside of it. It clearly has some sort of self-similarity, and a scaling factor of 2, but the number of self-similar objects needed to find the Hausdorff dimension is not as straightforward. By adjusting the formula to account for fractals containing fractals, we conjecture that a possible interpretation of the dimension of Pascal's Triangle mod 4 is

$$D = \frac{\log(3)}{\log(2)} + \frac{1}{16} \cdot \frac{\log(3)}{\log(2)}.$$

In order to calculate the dimension accurately, we use the Box Counting Method. By counting the pixels, a pattern emerges and we can get a formula for how many boxes exist in some p^n layers. These formulas, written as corollaries, lead to general formulas allowing any prime p with a fixed power n. We prove these general formulas using a form of induction where we show that plugging $n+1$ into the formula and continuing the observed pattern give the same result. Because this section focuses on Pascal's Triangle, we say "mod p^n" as an abbreviation of Pascal's Triangle mod p^n.

5.1. **Powers of Primes in Pascal's Triangle.**



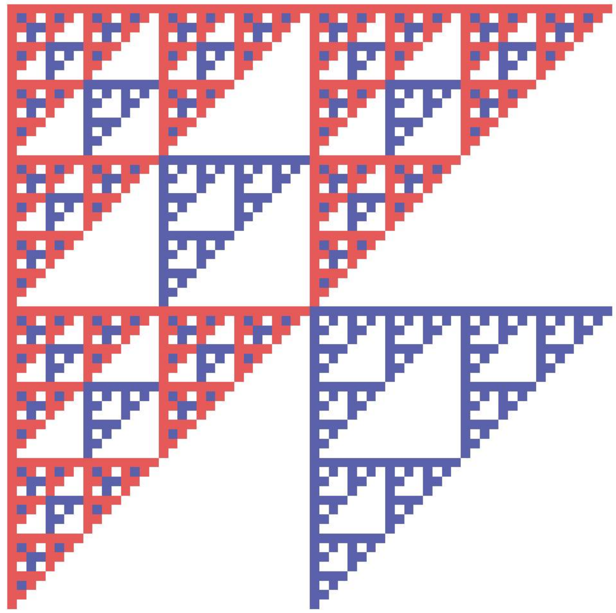
Mod 2 Triangle

When using the Box Counting Method on Pascal's Triangle mod 2, we find that each triangle has 2^n rows and 3^n pixels. As n increases, each iteration is equal to three times the previous triangle.

n	Pixels	Formula	n	Pixels	Formula
0	1	3^0	4	81	3^4
1	3	3^1	5	243	3^5
2	9	3^2	6	729	3^6
3	27	3^3	7	2187	3^7

We get the same result from multiplying by 3, and from plugging $n+1$ into 3^n. This means that each iteration is a power of 3. With this formula and the Box Counting Method, the fractal dimension of mod 2 is equal to

$$\frac{\log(3^n)}{\log(2^n)} = \frac{n\log(3)}{n\log(2)} = \frac{\log(3)}{\log(2)}.$$



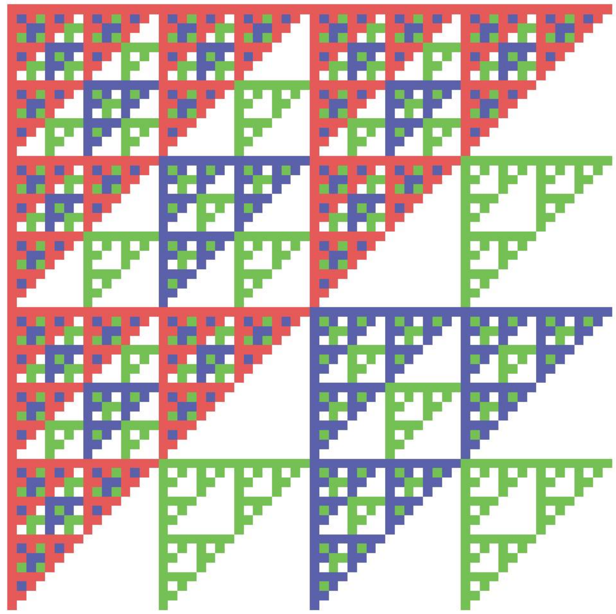
Mod 4 Triangle

The pattern in mod 4 continues with 3 times the previous iteration, plus a smaller mod 2 triangle added into the gap between them. Wanting a more accurate measurement of the dimension, we know that the image becomes a better approximation of the fractal if we let n go to infinity. Before we do this, however, we need to find a formula that gives the number of pixels in 2^n rows.

n	Pixels	Formula	n	Pixels	Formula
2	10	$3^2 + 1 \cdot 3^0$	5	351	$3^5 + 4 \cdot 3^3$
3	33	$3^3 + 2 \cdot 3^1$	6	1134	$3^6 + 5 \cdot 3^4$
4	108	$3^4 + 3 \cdot 3^2$	7	3645	$3^7 + 6 \cdot 3^5$

Corollary 5.1.1. *The formula for the number of pixels in a mod 4 triangle at* 2^n *rows is* $3^n + (n-1) \cdot 3^{n-2}$ *for all* $n \geq 1$.

This corollary and the ones in the following pages are specific cases of the general theorem in Section 5.2.



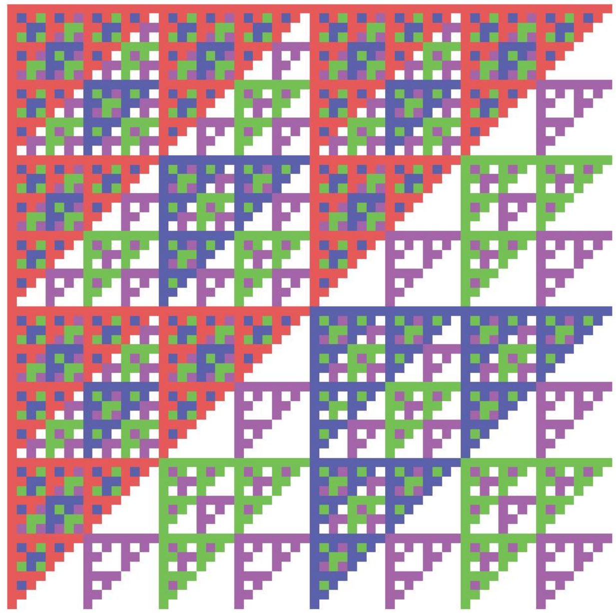
Mod 8 Triangle

In the mod 8 case, the overall pattern to find the next iteration is three times the current shape, plus a mod 4 triangle in the middle, plus three mod 2 triangles in the gaps around the mod 4 triangle.

n	Pixels	Formula	n	Pixels	Formula
3	36	$3^3 + 3 \cdot 3^1 + 0 \cdot 3^0$	6	1512	$3^6 + 9 \cdot 3^4 + 6 \cdot 3^2$
4	127	$3^4 + 5 \cdot 3^2 + 1 \cdot 3^1$	7	5130	$3^7 + 11 \cdot 3^5 + 10 \cdot 3^3$
5	441	$3^5 + 7 \cdot 3^3 + 3 \cdot 3^2$	8	17253	$3^8 + 13 \cdot 3^6 + 15 \cdot 3^4$

Corollary 5.1.2. *The formula for the number of pixels in a mod 8 triangle at 2^n rows is*

$$3^n + (2n-3) \cdot 3^{n-2} + \frac{(n-2)(n-3)}{2} \cdot 3^{n-4} \text{ for all } n \geq 2.$$



Mod 16 Triangle

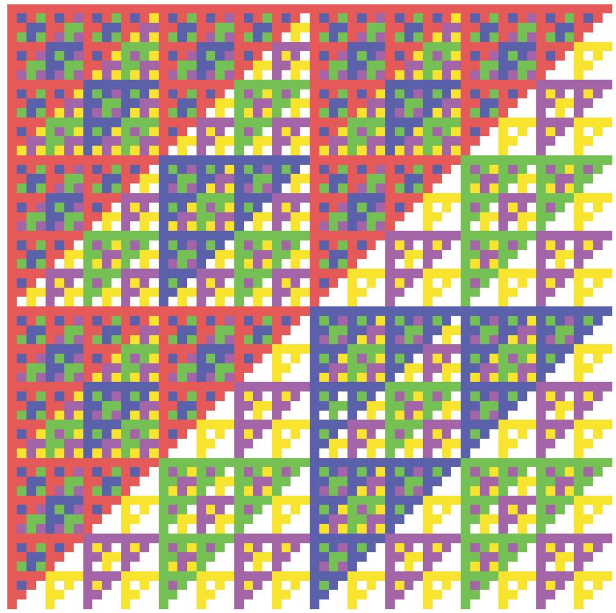
In the mod 16 case, the overall pattern for the next iteration is equal to three times the current iteration, plus one mod 8 triangle, three mod 4 triangles, and nine mod 2 triangles.

n	Pixels	Formula	n	Pixels	Formula
4	136	$3^4 + 6 \cdot 3^2 + 1 \cdot 3^0 + 0$	7	6438	$3^7 + 15 \cdot 3^5 + 22 \cdot 3^3 + 4 \cdot 3^1$
5	501	$3^5 + 9 \cdot 3^3 + 5 \cdot 3^1 + 0$	8	22608	$3^8 + 18 \cdot 3^6 + 35 \cdot 3^4 + 10 \cdot 3^2$
6	1810	$3^6 + 12 \cdot 3^4 + 12 \cdot 3^2 + 1 \cdot 3^0$	9	78543	$3^9 + 21 \cdot 3^7 + 51 \cdot 3^5 + 20 \cdot 3^3$

Corollary 5.1.3. *The formula for the number of pixels in a mod 16 triangle at 2^n rows is*

$$3^n + (3n-6) \cdot 3^{n-2} + \frac{(n-3)(3n-10)}{2} \cdot 3^{n-4} + \frac{(n-5)(n-4)(n-3)}{6} \cdot 3^{n-6}$$

for all $n \geq 3$.



Mod 32 Triangle

The mod 32 case continues the pattern of adding previous powers of 2. Each iteration is 3 times the previous, plus one mod 16, three mod 8's, nine mod 4's, and twenty seven mod 2's.

n	Pixels	Formula
5	528	$3^5 + 10 \cdot 3^3 + 5 \cdot 3^1 + 0 + 0$
6	1999	$3^6 + 14 \cdot 3^4 + 15 \cdot 3^2 + 1 \cdot 3^0 + 0$
7	7419	$3^7 + 18 \cdot 3^5 + 31 \cdot 3^3 + 7 \cdot 3^1 + 0$
8	27091	$3^8 + 22 \cdot 3^6 + 53 \cdot 3^4 + 22 \cdot 3^2 + 1 \cdot 3^0$
9	97593	$3^9 + 26 \cdot 3^7 + 81 \cdot 3^5 + 50 \cdot 3^3 + 5 \cdot 3$
10	347544	$3^{10} + 30 \cdot 3^8 + 115 \cdot 3^6 + 95 \cdot 3^4 + 15 \cdot 3^2$

This table also forms a base case for a proof by induction.

Theorem 5.1.4. *The formula for the number of pixels in a mod 32 triangle at 2^n rows is*

$$3^n + (4n-10) \cdot 3^{n-2} + (3n^2 - 23n + 45) \cdot 3^{n-4}$$

$$+\frac{(n-5)(n-4)(4n-21)}{6} \cdot 3^{n-6} + \frac{(n-7)(n-6)(n-5)(n-4)}{24} \cdot 3^{n-8}$$

for all $n \geq 4$.

Proof. For the base case, see the table above. Next, assume that the formula for the number of pixels in 2^n rows is

$$3^n + (4n-10)\cdot 3^{n-2} + (3n^2-23n+45)\cdot 3^{n-4}$$
$$+\frac{(n-5)(n-4)(4n-21)}{6}\cdot 3^{n-6} + \frac{(n-7)(n-6)(n-5)(n-4)}{24}\cdot 3^{n-8}.$$

We should get the same result from plugging $n+1$ into this formula, and from following the pattern of multiplying by 3 and and adding the other mod's. We get

$$3^{n+1} + (4n-6)\cdot 3^{n-1} + (3n^2-17n+25)\cdot 3^{n-3}$$
$$+\frac{(n-4)(n-3)(4n-17)}{6}\cdot 3^{n-5} + \frac{(n-6)(n-5)(n-4)(n-3)}{24}\cdot 3^{n-7}.$$

Next, we continue the pattern to see if they are equal. The pattern for mod 32 is 3 times the previous, plus one mod 16, three mod 8's, nine mod 4's, and twenty seven mod 2's.

$$3\cdot\Big(3^n + (4n-10)\cdot 3^{n-2} + (3n^2-23n+45)\cdot 3^{n-4}$$
$$+\frac{(n-5)(n-4)(4n-21)}{6}\cdot 3^{n-6} + \frac{(n-7)(n-6)(n-5)(n-4)}{24}\cdot 3^{n-8}\Big)$$
$$+3^{n-1} + (3n-9)\cdot 3^{n-3} + \frac{(n-4)(3n-13)}{2}\cdot 3^{n-5} + \frac{(n-6)(n-5)(n-4)}{6}\cdot 3^{n-7}$$
$$+3\left(3^{n-2} + (2n-7)\cdot 3^{n-4} + \frac{(n-4)(n-5)}{2}\cdot 3^{n-6}\right)$$
$$+3^2\left(3^{n-3} + (n-4)\cdot 3^{n-5}\right) + 3^3\left(3^{n-4}\right).$$

When simplified, this expression is equal to

$$3^{n+1} + (4n-6)\cdot 3^{n-1} + (3n^2-17n+25)\cdot 3^{n-3}$$
$$+\frac{(n-4)(n-3)(4n-17)}{6}\cdot 3^{n-5} + \frac{(n-6)(n-5)(n-4)(n-3)}{24}\cdot 3^{n-7}.$$

Since the two are equal, the formula is proven by induction. $\square$

The next table shows the numbers from the powers of 2.

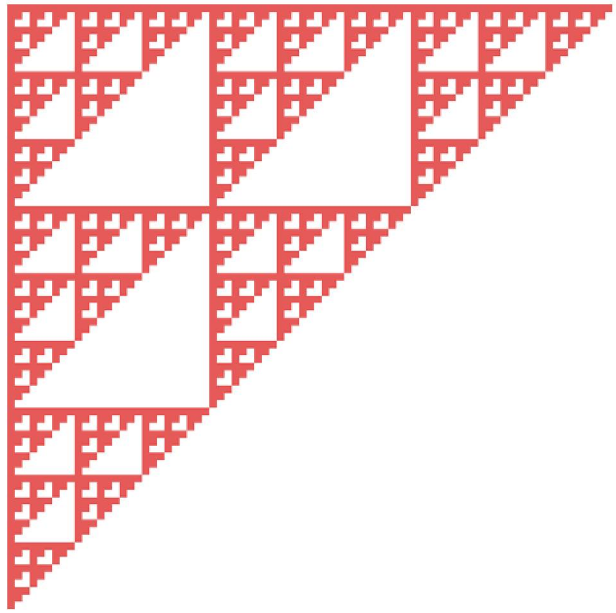
n	0	1	2	3	4	5	6	7	8	9	10
Mod 2	1	3	9	27	81	243	729	2187	6561	19683	59049
Mod 4	1	3	10	33	108	351	1134	3645	11664	37179	118098
Mod 8	1	3	10	36	127	441	1512	5130	17253	57591	190998
Mod 16	1	3	10	36	136	501	1810	6438	22608	78543	270378
Mod 32	1	3	10	36	136	528	1999	7419	27091	97593	347544

Note that every entry in the table starts with 3 times the left number. As we move down, the entries are also the sum of the numbers in an up and left diagonal with the property of one times one number, three times the next, nine times the next, and so on with increasing powers of 3. These powers of 3 correspond to the numbers of copies added with each iteration. Every iteration of a power of 2 triples the previous amount, then adds one triangle from the previous power of 2, three triangles from the next, and so on. For example, looking at the column where $n = 6$, we get the following results.

$n = 6$	Formula
729	$3 \cdot 243$
1134	$3 \cdot 351 + 81$
1512	$3 \cdot 441 + 108 + 3 \cdot 27$
1810	$3 \cdot 501 + 127 + 3 \cdot 33 + 9 \cdot 9$
1999	$3 \cdot 528 + 136 + 3 \cdot 36 + 9 \cdot 10 + 27 \cdot 3$

Additionally, every entry to the left of the down and right diagonal starting at mod 2 where $n = 2$ is a triangular number. This is because the space is limited when constructing these fractal triangles using a finite number of pixels.

Next, we examine powers of 3.



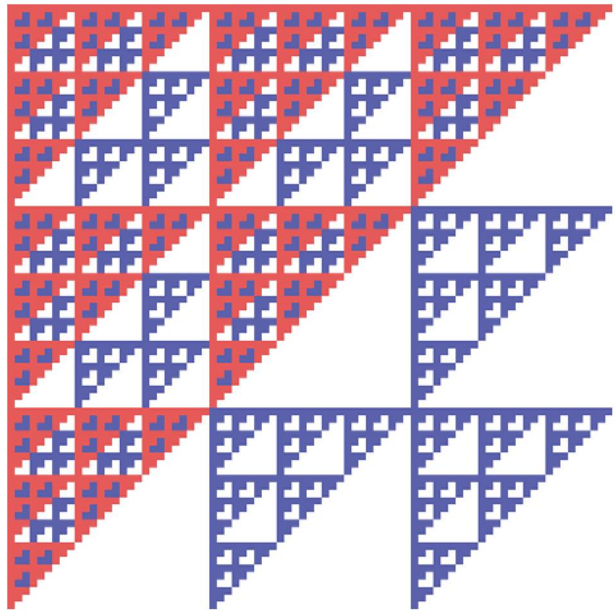
Mod 3 Triangle

For each of the powers of 3, the scaling factor is 3 with every iteration of the triangle containing 3^n rows. In the mod 3 case, every iteration contains 6^n pixels and the overall pattern is that the next iteration is equal to six times the previous.

n	Pixels	Formula	n	Pixels	Formula
0	1	6^0	4	1296	6^4
1	6	6^1	5	7776	6^5
2	36	6^2	6	46656	6^6
3	216	6^3	7	279936	6^7

The general formula for the number of pixels in 3^n rows is 6^n. When we plug in $n+1$ and continue the pattern by multiplying by six, we get the same result. Additionally, 6 is equal to the third triangular number $\frac{3(3+1)}{2}$. With the formula and the Box Counting Method, we find that the fractal dimension of mod 3 is equal to

$$\frac{\log(6^n)}{\log(3^n)} = \frac{n\log(6)}{n\log(3)} = \frac{\log(6)}{\log(3)}.$$



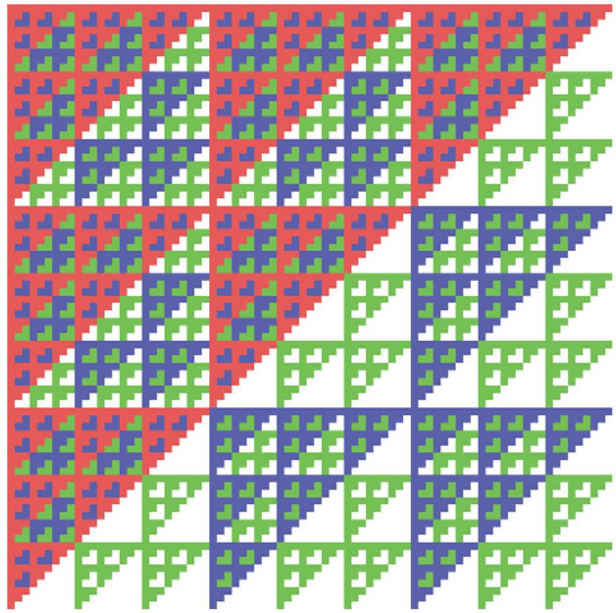
Mod 9 Triangle

In the mod 9 case, the pattern is that three smaller mod 3 triangles are added into each of the gaps for every iteration. One observation is that the additional triangles correspond exactly to the number and orientation of the gaps in the previous triangle. If we folded the image above diagonally to connect the top left corner to the bottom right corner, then the gaps in between the six triangles and the three additional blue triangles overlap. Looking back at the mod 2 case, the same is true. Based on this observation, we predict that the powers of 5 follow this trend and have 10 extra triangles added into each of the 10 gaps.

n	Pixels	Formula	n	Pixels	Formula
2	45	$6^2 + 1 \cdot 3^2 \cdot 6^0$	5	15552	$6^5 + 4 \cdot 3^2 \cdot 6^3$
3	324	$6^3 + 2 \cdot 3^2 \cdot 6^1$	6	104976	$6^6 + 5 \cdot 3^2 \cdot 6^4$
4	2268	$6^4 + 3 \cdot 3^2 \cdot 6^2$	7	699840	$6^7 + 6 \cdot 3^2 \cdot 6^5$

Corollary 5.1.5. *The formula for the number of pixels in a mod 9 triangle at 3^n rows is*

$$6^n + (n-1) \cdot 3^2 \cdot 6^{n-2} \text{ for all } n \geq 1.$$



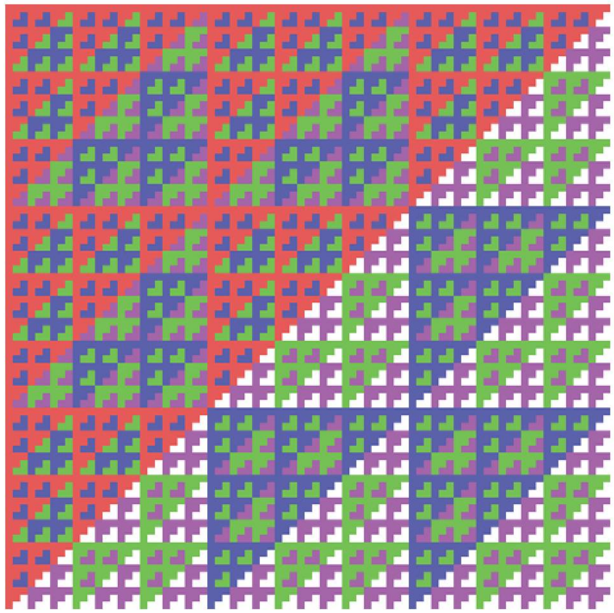
Mod 27 Triangle

In the mod 27 case, the pattern is that the mod 3 triangles have changed to mod 9 triangles since there are smaller mod 3 triangles added into each of their gaps. The remaining gaps are then filled with additional mod 3 triangles.

n	Pixels	Formula	n	Pixels	Formula
3	378	$6^3 + 3 \cdot 3^2 \cdot 6^1 + 0$	6	169128	$6^6 + 9 \cdot 3^2 \cdot 6^4 + 6 \cdot 3^4 \cdot 6^2$
4	2997	$6^4 + 5 \cdot 3^2 \cdot 6^2 + 1 \cdot 3^4 \cdot 6^0$	7	1224720	$6^7 + 11 \cdot 3^2 \cdot 6^5 + 10 \cdot 3^4 \cdot 6^3$
5	22842	$6^5 + 7 \cdot 3^2 \cdot 6^3 + 3 \cdot 3^4 \cdot 6^1$	8	8713008	$6^8 + 13 \cdot 3^2 \cdot 6^6 + 15 \cdot 3^4 \cdot 6^4$

Corollary 5.1.6. *The formula for the number of pixels in a mod 27 triangle at 3^n rows is*

$$6^n + (2n-3) \cdot 3^2 \cdot 6^{n-2} + \frac{(n-3)(n-2)}{2} \cdot 3^4 \cdot 6^{n-4} \textit{ for all } n \geq 2.$$



Mod 81 Triangle

In the mod 81 case, there are additional triangles added into every gap resulting in a sum of mod 27 triangles, mod 9 triangles, and mod 3 triangles.

n	Pixels	Formula
4	3321	$6^4 + 6 \cdot 3^2 \cdot 6^2 + 1 \cdot 3^4 \cdot 6^0 + 0$
5	27702	$6^5 + 9 \cdot 3^2 \cdot 6^3 + 5 \cdot 3^4 \cdot 6^1 + 0$
6	222345	$6^6 + 12 \cdot 3^2 \cdot 6^4 + 12 \cdot 3^4 \cdot 6^2 + 1 \cdot 3^6 \cdot 6^0$
7	1732104	$6^7 + 15 \cdot 3^2 \cdot 6^5 + 22 \cdot 3^4 \cdot 6^3 + 4 \cdot 3^6 \cdot 6^1$
8	13174488	$6^8 + 18 \cdot 3^2 \cdot 6^6 + 35 \cdot 3^4 \cdot 6^4 + 10 \cdot 3^6 \cdot 6^2$
9	98257536	$6^9 + 21 \cdot 3^2 \cdot 6^7 + 51 \cdot 3^4 \cdot 6^5 + 20 \cdot 9^3 \cdot 6^3$

Corollary 5.1.7. *The formula for the number of pixels in a mod 81 triangle at 3^n rows is*

$$6^n + (3n-6) \cdot 3^2 \cdot 6^{n-2} + \frac{(n-3)(3n-10)}{2} \cdot 3^4 \cdot 6^{n-4} + \frac{(n-5)(n-4)(n-3)}{6} \cdot 3^6 \cdot 6^{n-6}$$

for all $n \geq 3$.

The following table collects numbers from the powers of 3.

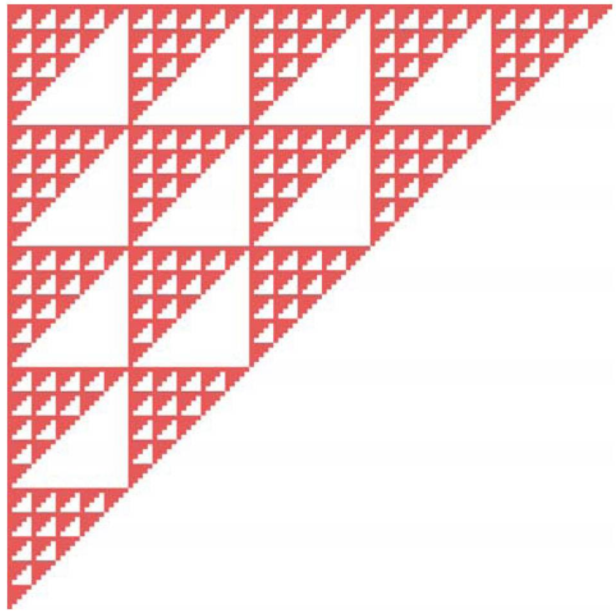
	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
Mod 3	1	6	36	216	1296	7776	46656	279936	1679616
Mod 9	1	6	45	324	2268	15552	104976	699840	4618944
Mod 27	1	6	45	378	2997	22842	169128	1224720	8713008
Mod 81	1	6	45	378	3321	27702	222345	1732104	13174488

Observe that every entry in the table starts with 6 times the left number, then adds the rest of the numbers in an up and left diagonal. The entries from this diagonal are multiplied by nine times one number, nine times six times the next, nine times thirty six times the next, and so on with increasing powers of 6. However, the nine stays constant for each multiplication. For example, looking at the column where $n = 6$, we get the following results.

$n=6$	Formula
46656	$6 \cdot 7776$
104976	$6 \cdot 15552 + 9 \cdot 1296$
169128	$6 \cdot 22842 + 9 \cdot 2268 + 9 \cdot 6 \cdot 216$
222345	$6 \cdot 27702 + 9 \cdot 2997 + 9 \cdot 6 \cdot 324 + 9 \cdot 6^2 \cdot 36$

Additionally, every entry to the left of the down and right diagonal starting at mod 3 where $n = 2$ is a triangular number. This is because of the limited space for fitting pixels into a triangle.

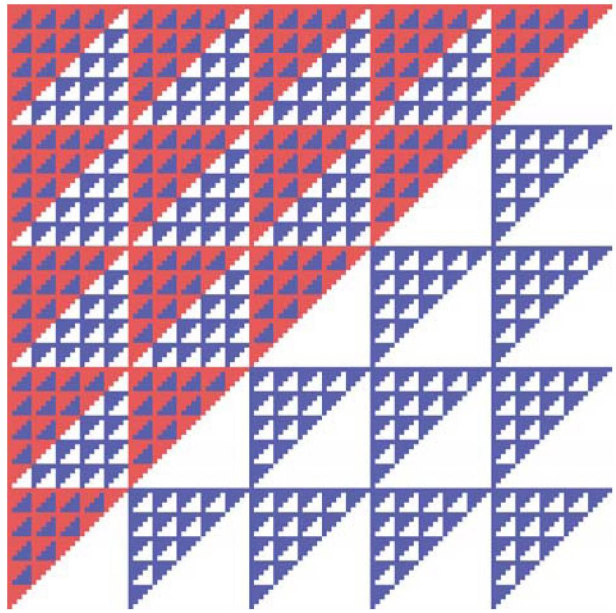
Next, we examine powers of 5.



Mod 5 Triangle

For every power of 5, the scaling factor is 5. In the mod 5 case, each iteration of the pattern takes the current number of pixels and multiplies it by 15. This is equal to the number of smaller triangles making up each triangle and it is also $15 = 1 + 2 + 3 + 4 + 5$. From the earlier powers of primes, $3 = 1 + 2$ was important in the formulas for mod 2, and $6 = 1+2+3$ was important for mod 3. Additionally, for powers of 3, the number 3 appeared in the formulas relating to the number of gaps. Since the powers of 2 only have one gap, the powers of 1 also appear, but without affecting the formulas. There is one pixel in the beginning and 15 after the first iteration of the pattern. Since the pattern generates 15 of each previous iteration, the total is always equal to 15^n. Using this formula and the Box Counting Method, we find that its fractal dimension is equal to

$$\frac{\log(15^n)}{\log(5^n)} = \frac{n\log(15)}{n\log(5)} = \frac{\log(15)}{\log(5)}.$$



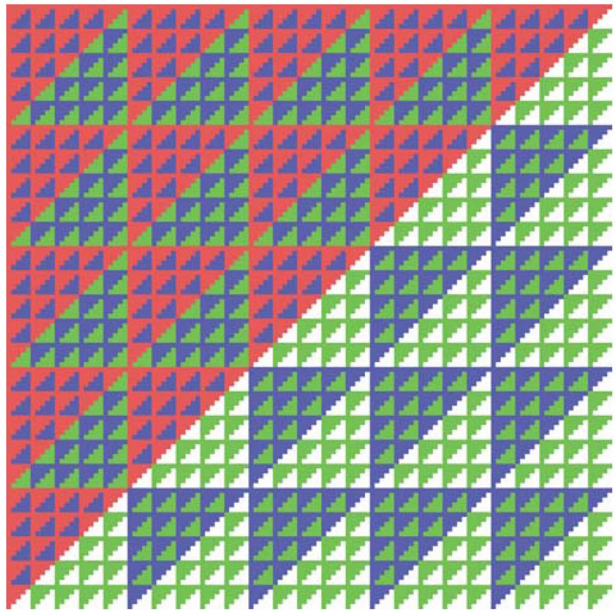
Mod 25 Triangle

In the mod 25 case, the pattern still has 15 times the number of pixels in the previous iteration, but then it adds $10 = 1 + 2 + 3 + 4$ new mod 5 triangles into each of the ten gaps. The positions of these new triangles also mirror the shape of the gaps in between the triangles in the previous iteration.

n	Pixels	Formula	n	Pixels	Formula
2	325	$15^2 + 1 \cdot 10^2 \cdot 15^0$	5	2109375	$15^5 + 4 \cdot 10^2 \cdot 15^3$
3	6375	$15^3 + 2 \cdot 10^2 \cdot 15^1$	6	36703125	$15^6 + 5 \cdot 10^2 \cdot 15^4$
4	118125	$15^4 + 3 \cdot 10^2 \cdot 15^2$	7	626484375	$15^7 + 6 \cdot 10^2 \cdot 15^5$

Corollary 5.1.8. *The formula for the number of pixels in a mod 25 triangle at 5^n rows is*

$$15^n + (n-1) \cdot 10^2 \cdot 15^{n-2} \textit{ for all } n \geq 1.$$



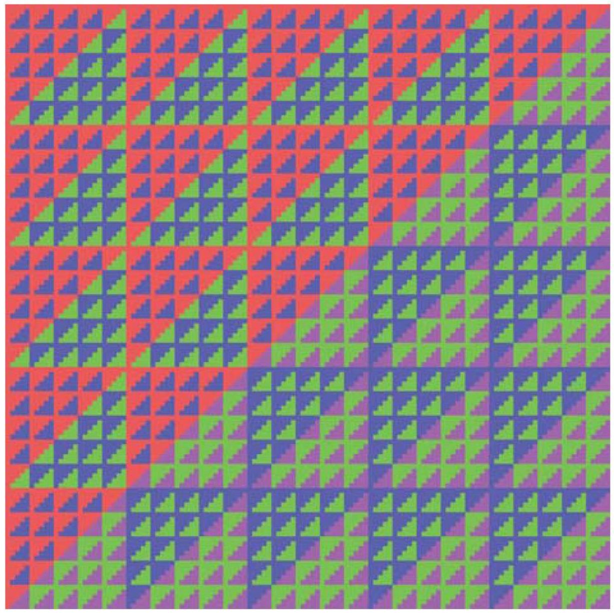
Mod 125 Triangle

In the mod 125 case, each iteration is made up of 15 times the previous iteration, however it changes the 10^2 mod 5 triangles into mod 25 triangles and adds $1500 = 10^2 \cdot 15$ smaller mod 5 triangles. The positions of these new triangles also mirror the shape of the gaps in between the triangles in the previous iteration.

n	Pixels	Formula	n	Pixels	Formula
3	7875	$15^3 + 3 \cdot 10^2 15^1$	6	70453125	$15^6 + 9 \cdot 10^2 15^4 + 6 \cdot 10^4 15^2$
4	173125	$15^4 + 5 \cdot 10^2 15^2 + 10^4 15^0$	7	134671875	$15^7 + 11 \cdot 10^2 15^5 + 10 \cdot 10^4 15^3$
5	3571875	$15^5 + 7 \cdot 10^2 15^3 + 3 \cdot 10^4 15$	8	24964453125	$15^8 + 13 \cdot 10^2 15^6 + 15 \cdot 10^4 15^4$

Corollary 5.1.9. *The formula for the number of pixels in a mod 125 triangle at 5^n rows is*

$$15^n + (2n-3) \cdot 10^2 \cdot 15^{n-2} + \frac{(n-3)(n-2)}{2} \cdot 10^4 \cdot 15^{n-4} \text{ for all } n \geq 2.$$



Mod 625 Triangle

In the mod 625 case, the pattern takes 15 times the previous iteration and changes the $100 = 10^2$ mod 25 triangles into mod 125 triangles, turns the $1500 = 10^2 \cdot 15$ mod 5 triangles into mod 25 triangles, and adds $22500 = 10^2 \cdot 15^2$ mod 5 triangles. The positions of these new triangles also mirror the shape of the gaps in between the triangles in the previous iteration.

n	Pixels	Formula
4	195625	$15^4 + 6 \cdot 10^2 \cdot 15^2 + 1 \cdot 10^4 \cdot 15^0$
5	4546875	$15^5 + 9 \cdot 10^2 \cdot 15^3 + 5 \cdot 10^4 \cdot 15$
6	100140625	$15^6 + 12 \cdot 10^2 \cdot 15^4 + 12 \cdot 10^4 \cdot 15^2 + 1 \cdot 10^6 \cdot 15^0$
7	2112421875	$15^7 + 15 \cdot 10^2 \cdot 15^5 + 22 \cdot 10^4 \cdot 15^3 + 4 \cdot 10^6 \cdot 15^1$
8	43034765625	$15^8 + 18 \cdot 10^2 \cdot 15^6 + 35 \cdot 10^4 \cdot 15^4 + 10 \cdot 10^6 \cdot 15^2$
9	852029296875	$15^9 + 21 \cdot 10^2 \cdot 15^7 + 51 \cdot 10^4 \cdot 15^5 + 20 \cdot 10^6 \cdot 15^3$

Corollary 5.1.10. *The number of pixels in a mod 625 triangle at 5^n rows is*

$$15^n + (3n-6)10^2 15^{n-2} + \left(\frac{(n-3)(3n-10)}{2}\right) 10^4 15^{n-4} + \left(\frac{(n-3)(n-4)(n-5)}{6}\right) 10^6 15^{n-6}$$

for all $n \geq 3$.

The following table contains numbers of pixels from the powers of 5.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
Mod 5	1	15	225	3375	50625	759375	11390625	170859375
Mod 25	1	15	325	6375	118125	2109375	36703125	626484375
Mod 125	1	15	325	7875	173125	3571875	70453125	1343671875
Mod 625	1	15	325	7875	195625	4546875	100140625	2112421875

Observe that every entry in the table starts with 15 times the left number, then adds the rest of the numbers in an up and left diagonal. The entries from this diagonal are multiplied by 100 times one number, 100 times 15 times the next, 100 times 15^2 times the next, and so on with increasing powers of 15. However, the 100 stays constant for each multiplication. For example, looking at the column where $n = 6$, we get the following results.

$n=6$	Formula
11390625	$15 \cdot 759375$
36703125	$15 \cdot 2109375 + 100 \cdot 50625$
70453125	$15 \cdot 3571875 + 100 \cdot 118125 + 100 \cdot 15 \cdot 3375$
100140625	$15 \cdot 4546875 + 100 \cdot 173125 + 100 \cdot 15 \cdot 6375 + 100 \cdot 15^2 \cdot 225$

Additionally, every entry to the left of the down and right diagonal starting at mod 5 where $n = 2$ is a triangular number. This is because of the limited space for fitting pixels into a triangle.

Next, we examine general patterns in the formulas.

5.2. **Patterns in Formulas.** From Section 4.2, we know that the mod p triangle always has $\frac{p(p+1)}{2} = T$ copies of itself. Since these copies are arranged in a triangle, the number of gaps between them is the previous triangular number equal to $\frac{p(p-1)}{2} = S$. In each of the following theorems, the proof is by induction, where the base case is formed from the tables in the earlier parts of this section.

Mod	Formula
4	$3^n + (n-1) \cdot 3^{n-2}$
9	$6^n + (n-1) \cdot 3^2 \cdot 6^{n-2}$
25	$15^n + (n-1) \cdot 10^2 \cdot 15^{n-2}$

When considering the mod p^2 triangle, there are additional mod p triangles added into the gaps. There are $\frac{p(p-1)}{2}$ gaps and each one contains $\frac{p(p-1)}{2}$ additional triangles corresponding to the number of gaps in the smaller copy. This results in S^2 copies of the mod p triangles.

Theorem 5.2.1. *The formula for the number of pixels in a mod p^2 triangle at p^n rows is*

$$T^n + (n-1) \cdot S^2 \cdot T^{n-2}.$$

Proof. We check to see if we get the same result from plugging in $n+1$, and from continuing the pattern. Plugging in $n+1$, we get

$$T^{n+1} + n \cdot S^2 \cdot T^{n-1}.$$

Continuing the pattern, we get

$$T(T^n + (n-1) \cdot S^2 \cdot T^{n-2}) + S^2(T^{n-1}) = T^{n+1} + n \cdot S^2 \cdot T^{n-1}.$$

The two results are equal, which proves the formula by induction. $\square$

Corollary 5.2.2. *The fractal dimension of a mod p^2 triangle is the same as mod p.*

Proof. The fractal dimension of Pascal's Triangle mod p^2 is

$$\lim_{n\to\infty} \frac{\log(T^n + (n-1) \cdot S^2 \cdot T^{n-2})}{\log(p^n)} = \lim_{n\to\infty} \frac{n\log(T) + \log(1 + (n-1) \cdot S^2 \cdot T^{-2})}{n\log(p)} = \frac{\log(T)}{\log(p)}.$$

$\square$

Mod	Formula
8	$3^n + (2n-3)\cdot 3^{n-2} + \frac{(n-2)(n-3)}{2}\cdot 3^{n-4}$
27	$6^n + (2n-3)\cdot 3^2\cdot 6^{n-2} + \frac{(n-2)(n-3)}{2}\cdot 3^4\cdot 6^{n-4}$
25	$15^n + (2n-3)\cdot 10^2\cdot 15^{n-2} + \frac{(n-2)(n-3)}{2}\cdot 10^4\cdot 15^{n-4}$

When considering the mod p^3 triangle, the additional S^2 mod p triangles become mod p^2 triangles, and the remaining gaps have $S^2\cdot T$ mod p triangles added.

Theorem 5.2.3. *The formula for the number of pixels in a mod p^3 triangle at p^n rows is*

$$T^n + (2n-3)\cdot S^2\cdot T^{n-2} + \frac{(n-3)(n-2)}{2}\cdot S^4\cdot T^{n-4}.$$

Proof. We check to see if we get the same result from plugging in $n+1$, and from continuing the pattern. Plugging in $n+1$, we get

$$T^{n+1} + (2n-1)\cdot S^2\cdot T^{n-1} + \frac{(n-2)(n-1)}{2}\cdot S^4\cdot T^{n-3}.$$

When we continue the pattern, we get

$$T(T^n+(2n-3)\cdot S^2\cdot T^{n-2}+\frac{(n-3)(n-2)}{2}\cdot S^4\cdot T^{n-4})+S^2(T^{n-1}+(n-2)\cdot S^2\cdot T^{n-3})+S^2\cdot T\cdot(T^{n-2}).$$

When simplified, this matches the formula with $n+1$, as required to prove the formula. $\square$

Corollary 5.2.4. *The fractal dimension of a mod p^3 triangle is the same as mod p.*

Proof. The fractal dimension of a mod p^3 triangle is

$$\lim_{n\to\infty} \frac{\log(T^n + (2n-3)\cdot S^2\cdot T^{n-2} + \frac{(n-3)(n-2)}{2}\cdot S^4\cdot T^{n-4})}{\log(p^n)}$$

$$= \lim_{n\to\infty} \frac{n\log(T) + \log(1 + (2n-3)\cdot S^2\cdot T^{-2} + \frac{(n-3)(n-2)}{2}\cdot S^4\cdot T^{-4})}{n\log(p)} = \frac{\log(T)}{\log(p)}.$$

$\square$

Mod	Formula
16	$3^n + (3n-6)\cdot 3^{n-2} + \frac{(n-3)(3n-10)}{2}\cdot 3^{n-4} + \frac{(n-5)(n-4)(n-3)}{6}\cdot 3^{n-6}$
81	$6^n + (3n-6)\cdot 3^2\cdot 6^{n-2} + \frac{(n-3)(3n-10)}{2}\cdot 3^4\cdot 6^{n-4} + \frac{(n-5)(n-4)(n-3)}{6}\cdot 3^6\cdot 6^{n-6}$
125	$15^n + (3n-6)\cdot 10^2\cdot 15^{n-2} + \frac{(n-3)(3n-10)}{2}\cdot 10^4\cdot 15^{n-4} + \frac{(n-5)(n-4)(n-3)}{6}\cdot 10^6\cdot 15^{n-6}$

For the mod p^4 triangles, we have more triangles added to the gaps.

Theorem 5.2.5. *The formula for the number of pixels in a mod p^4 triangle at p^n rows is*

$$T^n + (3n-6)\cdot S^2\cdot T^{n-2} + \frac{(n-3)(3n-10)}{2}\cdot S^4\cdot T^{n-4} + \frac{(n-5)(n-4)(n-3)}{6}\cdot S^6\cdot T^{n-6}.$$

Proof. Plugging in $n+1$ gives us

$$T^{n+1} + (3n-3)\cdot S^2\cdot T^{n-1} + \frac{(n-2)(3n-7)}{2}\cdot S^4\cdot T^{n-3} + \frac{(n-4)(n-3)(n-2)}{6}\cdot S^6\cdot T^{n-5}.$$

Continuing the pattern, we get

$$T^{n+1} + (3n-6)\cdot S^2\cdot T^{n-1} + \frac{(n-3)(3n-10)}{2}\cdot S^4\cdot T^{n-3} + \frac{(n-5)(n-4)(n-3)}{6}\cdot S^6\cdot T^{n-5}$$

$$+S^2(T^{n-1} + (2n-5)\cdot S^2\cdot T^{n-3} + \frac{(n-4)(n-3)}{2}\cdot S^4\cdot T^{n-5})$$

$$+S^2\cdot T(T^{n-2} + (n-3)\cdot S^2\cdot T^{n-4}) + S^2\cdot T^2(T^{n-3}).$$

Simplifying this gives us the result from $n+1$, which completes the induction. □

Corollary 5.2.6. *The fractal dimension of a mod p^4 triangle is the same as mod p.*

Proof. The fractal dimension of a mod p^4 triangle is

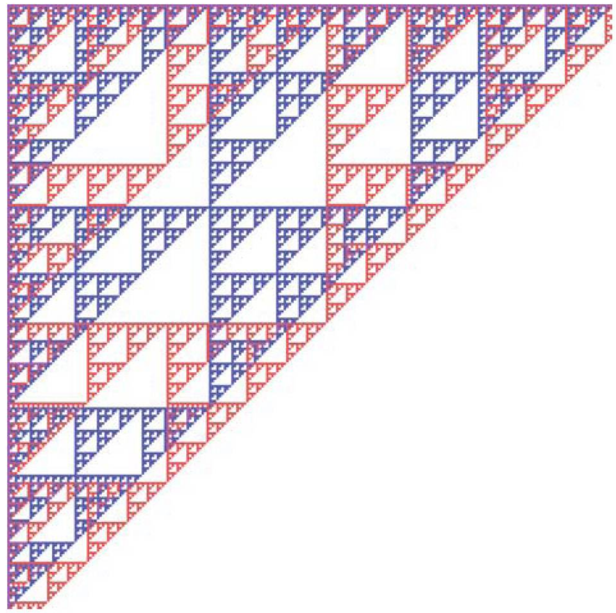
$$\lim_{n\to\infty} \frac{\log(T^n + (3n-6)S^2T^{n-2} + \frac{(n-3)(3n-10)}{2}S^4T^{n-4} + \frac{(n-5)(n-4)(n-3)}{6}S^6T^{n-6})}{\log(p^n)}$$

$$= \lim_{n\to\infty} \frac{n\log(T) + \log(1 + (3n-6)S^2T^{-2} + \frac{(n-3)(3n-10)}{2}S^4T^{-4} + \frac{(n-5)(n-4)(n-3)}{6}S^6T^{-6})}{n\log(p)}$$

$$= \frac{\log(T)}{\log(p)}$$

□

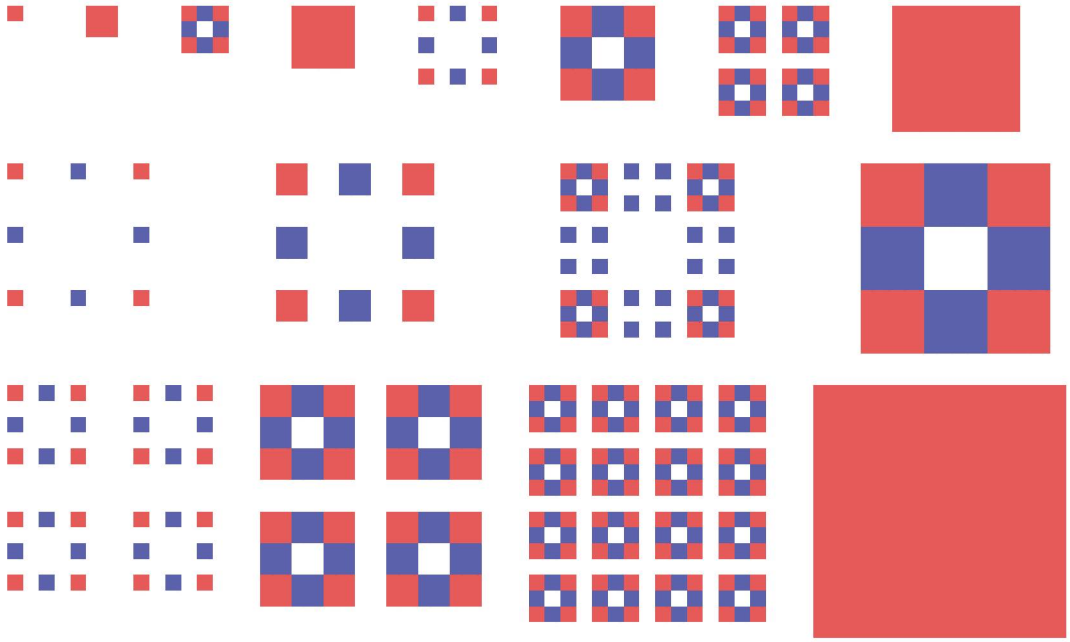
5.3. **Mod 6 and Conclusions.**



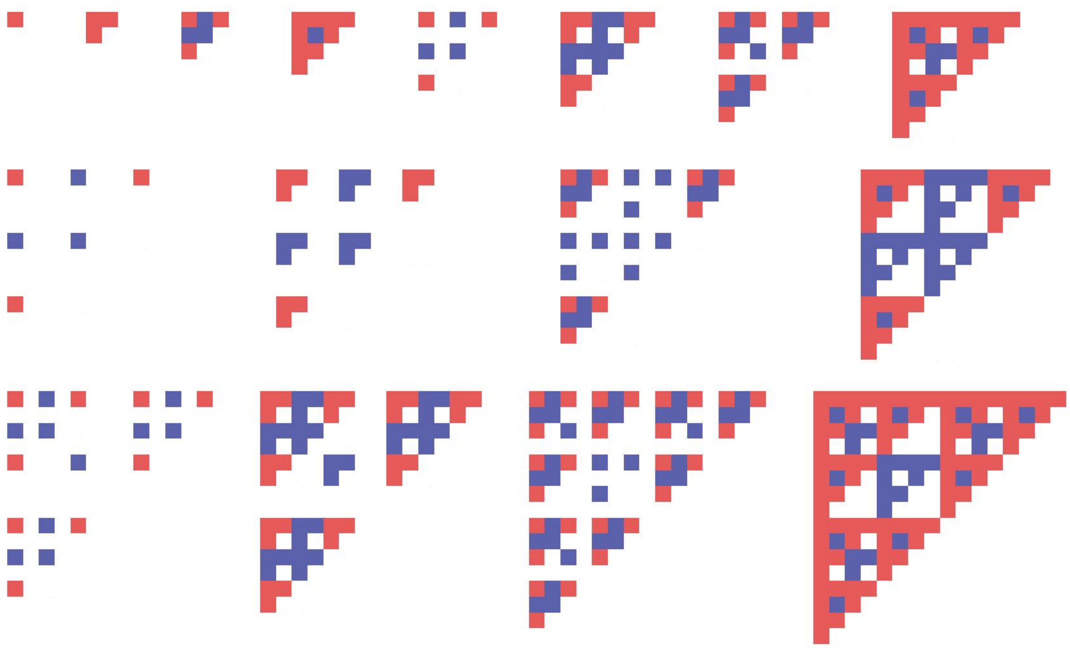
Mod 6 "Triangle"

The pattern of the nonzero entries in Pascal's Triangle mod 6 is actually the pattern from mod 2 overlapping with the pattern from mod 3, which follows from the Chinese Remainder Theorem. Since a power of 2 (other than $2^0 = 1$) is never equal to a power of 3 according the Fundamental Theorem of Arithmetic, these patterns never overlap onto the same row and create a triangular object. This is because the fractals in Pascal's Triangle mod 2 "end" at powers of 2 to make self-similar objects, and for mod 3, the fractals "end" at powers of 3. This means there is no original "self" to which the pattern can be similar. In the same manner, all other moduli other than prime powers do not form self-similar patterns. Without self-similarity, these patterns do not meet the definition of a fractal.

In conclusion, Pascal's Pyramidions and Simplexes have formulas that calculate fractal dimensions corresponding to any dimension $d \in \mathbb{N}$ and any prime number p. Additionally, the powers of primes up to p^4 form fractals in Pascal's Triangle with the same fractal dimension as the primes. From this observation, we conjecture that the fractal dimension of the fractal generated by Pascal's Triangle mod p^n is equal to the fractal dimension of Pascal's Triangle mod p. One topic for future research is to prove this conjecture for any n. Another topic is to study the powers of primes in the generalizations of Pascal's Triangle.



Mod 4 Pyramid



Mod 4 Tetrahedron

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